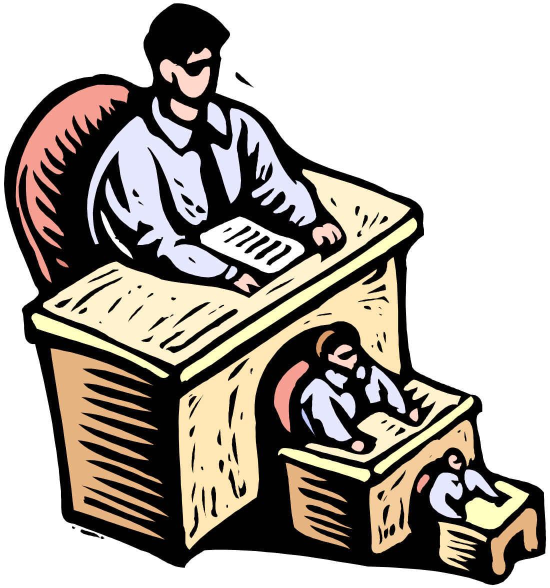
Recursion



The Recursion Pattern

- **Recursion**: when a method calls itself
- Classic example--the factorial function:
  - $n! = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot (n-1) \cdot n$
- Recursive definition:

$$f(n) = \begin{cases} 1 & \text{if } n = 0 \\ n \cdot f(n-1) & \textit{else} \end{cases}$$

- As a Python method:

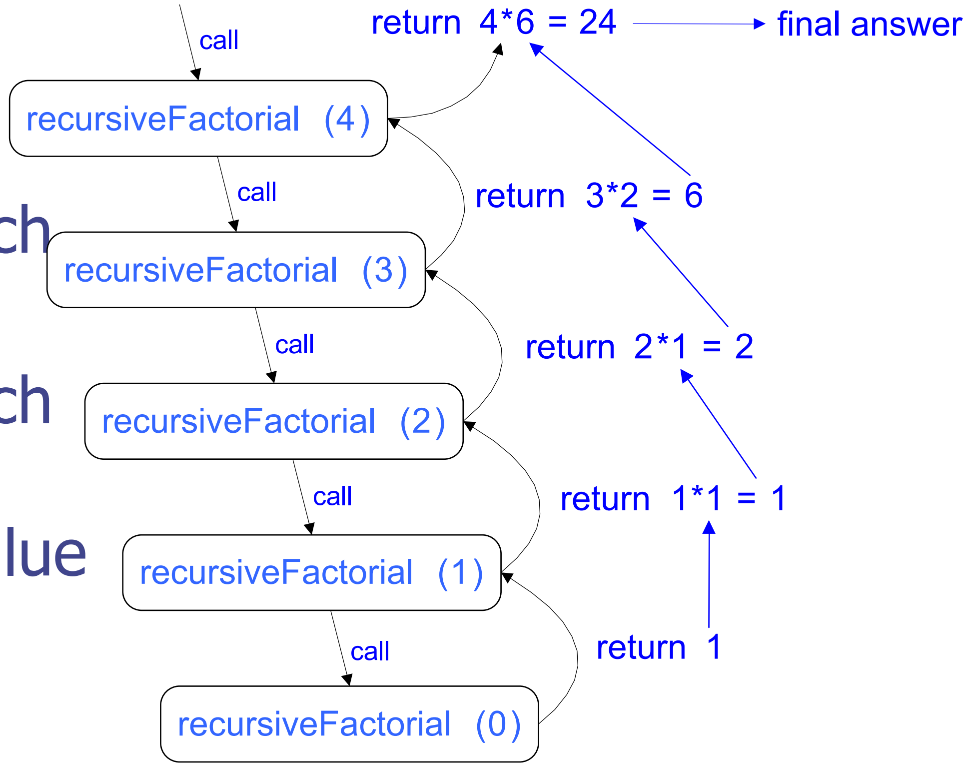
```
def factorial(n):
  if n == 0:
    return 1
  else:
    return n * factorial(n−1)
```

Content of a Recursive Method

- Base case(s)
  - Values of the input variables for which we perform no recursive calls are called base cases (there should be at least one base case).
  - Every possible chain of recursive calls must eventually reach a base case.
- Recursive calls
  - Calls to the current method.
  - Each recursive call should be defined so that it makes progress towards a base case.

Visualizing Recursion

- Recursion trace
  - A box for each recursive call
  - An arrow from each caller to callee
  - An arrow from each callee to caller showing return value
- Example



Example: English Ruler

- Print the ticks and numbers like an English ruler:

```
---- 0
-
--
-
---
-
--
-
---- 1
-
--
-
---
-
--
-
---- 2
```

```
----- 0
-
--
-
---
-
--
-
----
-
--
-
---
-
--
-
----- 1
```

```
--- 0
-
--
-
--- 1
-
--
-
--- 2
-
--
-
--- 3
```

Slide by Matt Stallmann included with permission.

Using Recursion

drawTicks(length)

Input: length of a 'tick'

Output: ruler with tick of the given length in the middle and smaller rulers on either side

```
---- 0          ----- 0          --- 0
-               -                -
--              --               --
-               -                -
---             ---              --- 1
-               -                -
--              --               --
-               -                -
---- 1          ----             --- 2
-               -                -
--              --               --
-               -                -
---             ---              --- 3
-               -
--              --
-               -
---- 2          ----- 1
```

drawTicks(length)

if(length > 0) then

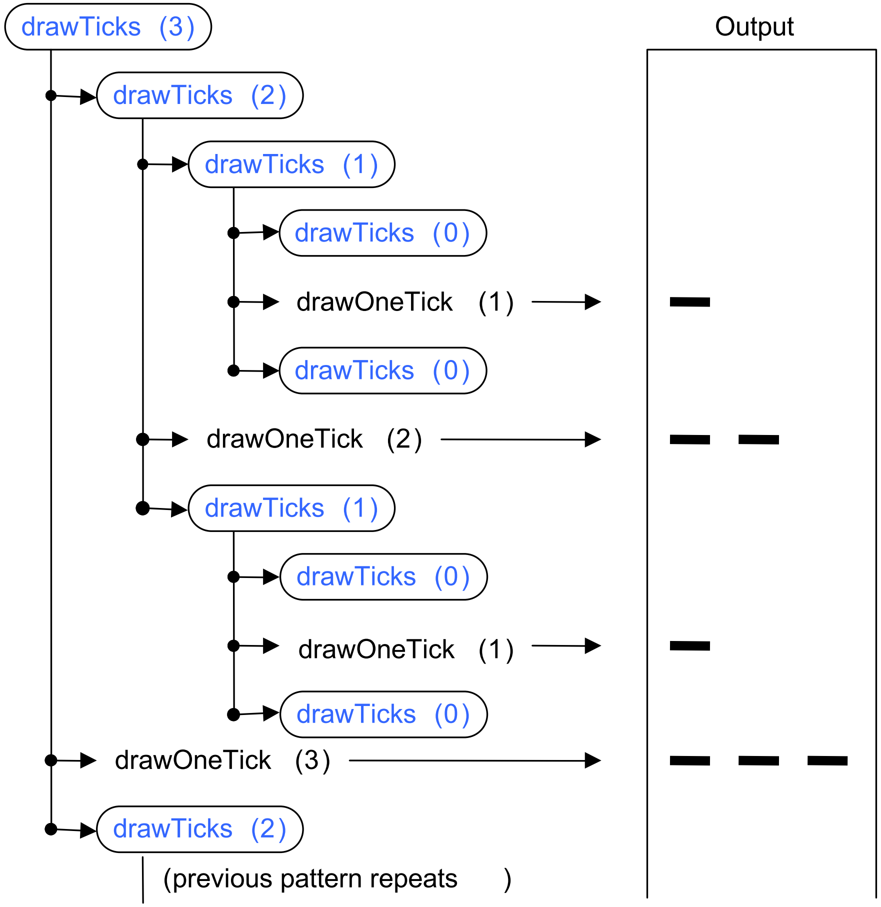
drawTicks(length – 1)

draw tick of the given length

drawTicks(length – 1)

Recursive Drawing Method

- The drawing method is based on the following recursive definition
- An interval with a central tick length L ≥ 1 consists of:
  - An interval with a central tick length L–1
  - An single tick of length L
  - An interval with a central tick length L–1



A Recursive Method for Drawing Ticks on an English Ruler

```
def draw_line(tick_length, tick_label=''):
  """Draw one line with given tick length (followed by optional label)."""
  line = '-' * tick_length
  if tick_label:
    line += ' ' + tick_label
  print(line)

def draw_interval(center_length):
  """Draw tick interval based upon a central tick length."""
  if center_length > 0:                # stop when length drops to 0
    draw_interval(center_length - 1)   # recursively draw top ticks
    draw_line(center_length)           # draw center tick
    draw_interval(center_length - 1)   # recursively draw bottom ticks

def draw_ruler(num_inches, major_length):
  """Draw English ruler with given number of inches, major tick length."""
  draw_line(major_length, '0')         # draw inch 0 line
  for j in range(1, 1 + num_inches):
    draw_interval(major_length - 1)    # draw interior ticks for inch
    draw_line(major_length, str(j))    # draw inch j line and label
```

Note the two recursive calls

Binary Search

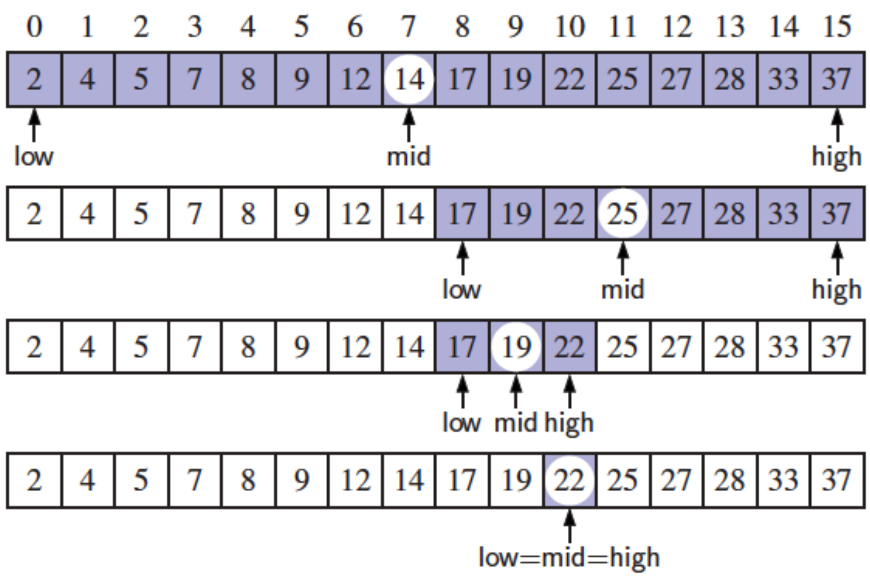
- Search for an integer, target, in an ordered list.

```
def binary_search(data, target, low, high):
  """Return True if target is found in indicated portion of a Python list.

  The search only considers the portion from data[low] to data[high] inclusive.
  """
  if low > high:
    return False                          # interval is empty; no match
  else:
    mid = (low + high) // 2
    if target == data[mid]:               # found a match
      return True
    elif target < data[mid]:
      # recur on the portion left of the middle
      return binary_search(data, target, low, mid − 1)
    else:
      # recur on the portion right of the middle
      return binary_search(data, target, mid + 1, high)
```

Visualizing Binary Search

- We consider three cases:
  - If the target equals data[mid], then we have found the target.
  - If target < data[mid], then we recur on the first half of the sequence.
  - If target > data[mid], then we recur on the second half of the sequence.



Analyzing Binary Search

- Runs in O(log n) time.
  - The remaining portion of the list is of size high – low + 1.
  - After one comparison, this becomes one of the following:

$$(\mathsf{mid}-1)-\mathsf{low}+1=\left\lfloor\frac{\mathsf{low}+\mathsf{high}}{2}\right\rfloor-\mathsf{low}\leq\frac{\mathsf{high}-\mathsf{low}+1}{2}$$

$$\mathsf{high}-(\mathsf{mid}+1)+1=\mathsf{high}-\left\lfloor\frac{\mathsf{low}+\mathsf{high}}{2}\right\rfloor\leq\frac{\mathsf{high}-\mathsf{low}+1}{2}.$$

  - Thus, each recursive call divides the search region in half; hence, there can be at most log n levels.

Linear Recursion

- Test for base cases
  - Begin by testing for a set of base cases (there should be at least one).
  - Every possible chain of recursive calls must eventually reach a base case, and the handling of each base case should not use recursion.
- Recur once
  - Perform a single recursive call
  - This step may have a test that decides which of several possible recursive calls to make, but it should ultimately make just one of these calls
  - Define each possible recursive call so that it makes progress towards a base case.

Example of Linear Recursion

Algorithm LinearSum(*A*, *n*):

Input:

A integer array *A* and an integer *n* = 1, such that *A* has at least *n* elements

Output:

The sum of the first *n* integers in *A*

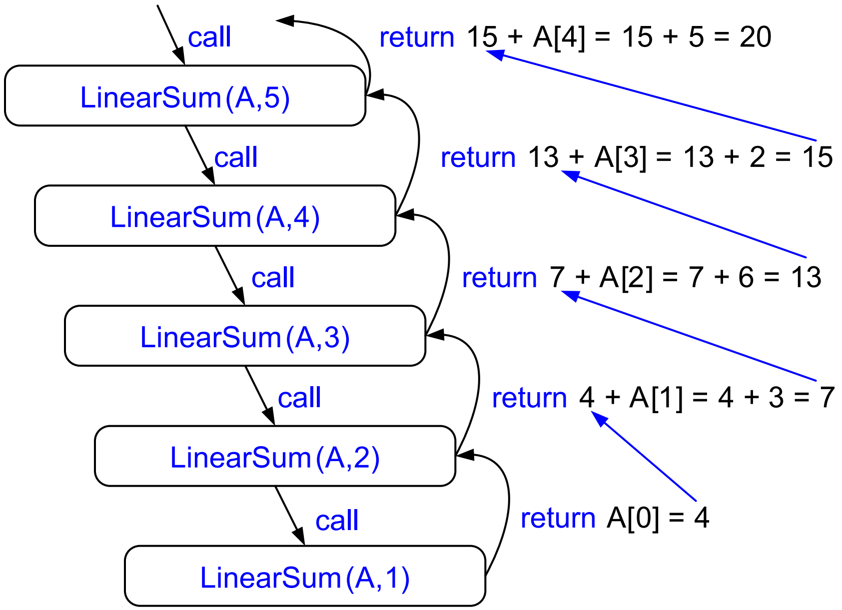
if *n* = 1 **then**

  return *A*[0]

else

  return LinearSum(*A*, *n* - 1) + *A*[*n* - 1]

Example recursion trace:



Reversing an Array

Algorithm ReverseArray(A, i, j):

Input: An array A and nonnegative integer indices i and j

Output: The reversal of the elements in A starting at index i and ending at j

if $i < j$ **then**

Swap $A[i]$ and $A[j]$

ReverseArray(A, $i + 1$, $j - 1$)

return

Defining Arguments for Recursion

- In creating recursive methods, it is important to define the methods in ways that facilitate recursion.
- This sometimes requires we define additional paramaters that are passed to the method.
- For example, we defined the array reversal method as ReverseArray(*A*, *i*, *j*), not ReverseArray(*A*).
- Python version:

```
def reverse(S, start, stop):
  """Reverse elements in implicit slice S[start:stop]."""
  if start < stop − 1:                      # if at least 2 elements:
    S[start], S[stop−1] = S[stop−1], S[start]   # swap first and last
    reverse(S, start+1, stop−1)              # recur on rest
```

Computing Powers

- The power function, $p(x,n)=x^n$, can be defined recursively:

$$p(x,n) = \begin{cases} 1 & \text{if } n = 0 \\ x \cdot p(x, n-1) & \text{else} \end{cases}$$

- This leads to an power function that runs in O(n) time (for we make n recursive calls).
- We can do better than this, however.

Recursive Squaring

- We can derive a more efficient linearly recursive algorithm by using repeated squaring:

$$p(x,n) = \begin{cases} 1 & \text{if } x = 0 \\ x \cdot p(x,(n-1)/2)^2 & \text{if } x > 0 \text{ is odd} \\ p(x,n/2)^2 & \text{if } x > 0 \text{ is even} \end{cases}$$

- For example,

$2^4 = 2^{(4/2)2} = (2^{4/2})^2 = (2^2)^2 = 4^2 = 16$

$2^5 = 2^{1+(4/2)2} = 2(2^{4/2})^2 = 2(2^2)^2 = 2(4^2) = 32$

$2^6 = 2^{(6/2)2} = (2^{6/2})^2 = (2^3)^2 = 8^2 = 64$

$2^7 = 2^{1+(6/2)2} = 2(2^{6/2})^2 = 2(2^3)^2 = 2(8^2) = 128.$

Recursive Squaring Method

Algorithm Power(x, n):

Input: A number x and integer $n = 0$

Output: The value x^n

if $n = 0$ **then**

  return 1

if n is odd **then**

  $y = \text{Power}(x, (n - 1)/2)$

  return $x \cdot y \cdot y$

else

  $y = \text{Power}(x, n/2)$

  return $y \cdot y$

Analysis

Algorithm Power(x, n):
    Input: A number x and integer $n = 0$
    Output: The value x^n
  if $n = 0$ **then**
      return 1
  if n is odd **then**
      y = Power(x, (n - 1)/ 2)
      return x ' y ' y
  else
      y = Power(x, n/ 2)
      return y ' y

Each time we make a recursive call we halve the value of n; hence, we make log n recursive calls. That is, this method runs in O(log n) time.

It is important that we use a variable twice here rather than calling the method twice.

Tail Recursion

- Tail recursion occurs when a linearly recursive method makes its recursive call as its last step.
- The array reversal method is an example.
- Such methods can be easily converted to non-recursive methods (which saves on some resources).
- Example:

Algorithm IterativeReverseArray(A, i, j):

Input: An array A and nonnegative integer indices i and j

Output: The reversal of the elements in A starting at index i and ending at j

while $i < j$ **do**

Swap $A[i]$ and $A[j]$

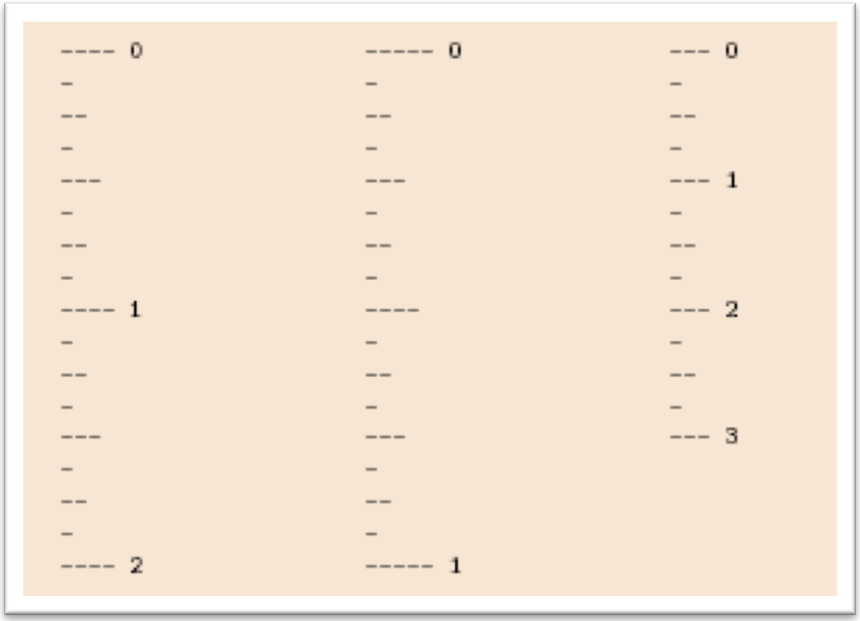
$i = i + 1$

$j = j - 1$

return

Binary Recursion

- Binary recursion occurs whenever there are **two** recursive calls for each non-base case.
- Example from before: the DrawTicks method for drawing ticks on an English ruler.



```
---- 0
-
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-
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-
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-
---- 1
-
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-
---
-
--
-
---- 2
```

```
----- 0
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---
-
--
-
----
-
--
-
---
-
--
-
----- 1
```

```
--- 0
-
--
-
--- 1
-
--
-
--- 2
-
--
-
--- 3
```

Another Binary Recusive Method

- Problem: add all the numbers in an integer array A:

  Algorithm BinarySum(A, i, n):

  Input: An array A and integers i and n

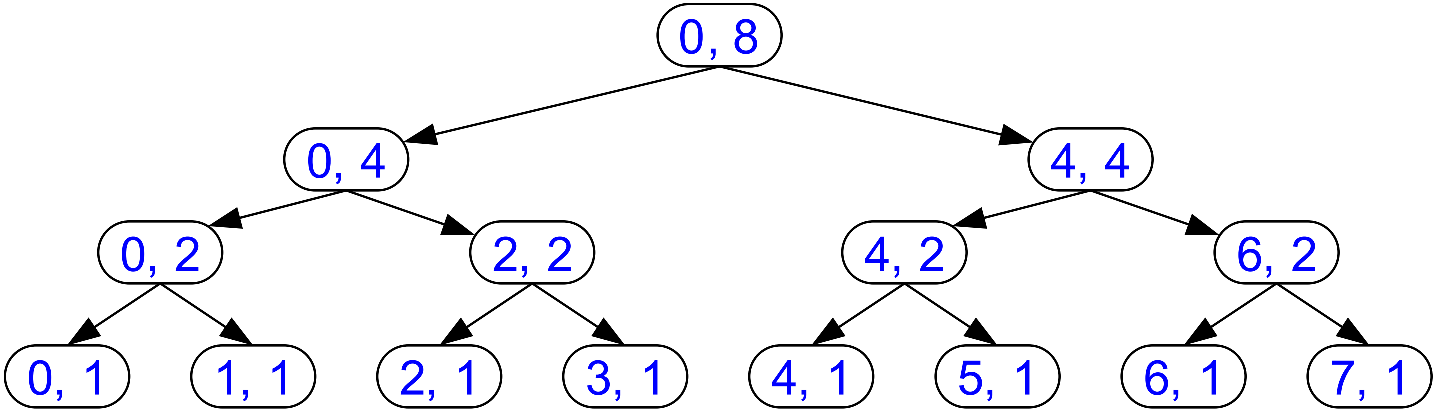
  Output: The sum of the n integers in A starting at index i

  if $n = 1$ **then**

  return $A[i]$

  return BinarySum(A, i, $n/2$) + BinarySum(A, $i + n/2$, $n/2$)

- **Example trace:**



Computing Fibonacci Numbers

- Fibonacci numbers are defined recursively:

$$F_0 = 0$$
$$F_1 = 1$$
$$F_i = F_{i-1} + F_{i-2} \quad \text{for } i > 1.$$

- Recursive algorithm (first attempt):

```
Algorithm BinaryFib(k):
   Input: Nonnegative integer k
   Output: The kth Fibonacci number F_k
  if k = 1 then
   return k
  else
   return BinaryFib(k - 1) + BinaryFib(k - 2)
```

Analysis

- Let n_k be the number of recursive calls by BinaryFib(k)
  - $n_0 = 1$
  - $n_1 = 1$
  - $n_2 = n_1 + n_0 + 1 = 1 + 1 + 1 = 3$
  - $n_3 = n_2 + n_1 + 1 = 3 + 1 + 1 = 5$
  - $n_4 = n_3 + n_2 + 1 = 5 + 3 + 1 = 9$
  - $n_5 = n_4 + n_3 + 1 = 9 + 5 + 1 = 15$
  - $n_6 = n_5 + n_4 + 1 = 15 + 9 + 1 = 25$
  - $n_7 = n_6 + n_5 + 1 = 25 + 15 + 1 = 41$
  - $n_8 = n_7 + n_6 + 1 = 41 + 25 + 1 = 67.$
- Note that n_k at least doubles every other time
- That is, $n_k > 2^{k/2}$. It is exponential!

A Better Fibonacci Algorithm

- Use linear recursion instead

```
Algorithm LinearFibonacci(k):
  Input: A nonnegative integer k
  Output: Pair of Fibonacci numbers (Fk , Fk−1)
  if k = 1 then
      return (k, 0)
  else
      (i,  j)  =  LinearFibonacci(k − 1)
      return (i +j, i)
```

- LinearFibonacci makes $k-1$ recursive calls

Multiple Recursion

- Motivating example:
  - summation puzzles
    - *pot* + *pan* = *bib*
    - *dog* + *cat* = *pig*
    - *boy* + *girl* = *baby*
- Multiple recursion:
  - makes potentially many recursive calls
  - not just one or two

Algorithm for Multiple Recursion

Algorithm PuzzleSolve(k,S,U):

Input: Integer k, sequence S, and set U (universe of elements to test)

Output: Enumeration of all k-length extensions to S using elements in U without repetitions

```
for all e in U do
    Remove e from U        {e is now being used}
    Add e to the end of S
    if k = 1 then
        Test whether S is a configuration that solves the puzzle
        if S solves the puzzle then
            return "Solution found: " S
    else
        PuzzleSolve(k - 1, S,U)
    Add e back to U        {e is now unused}
    Remove e from the end of S
```

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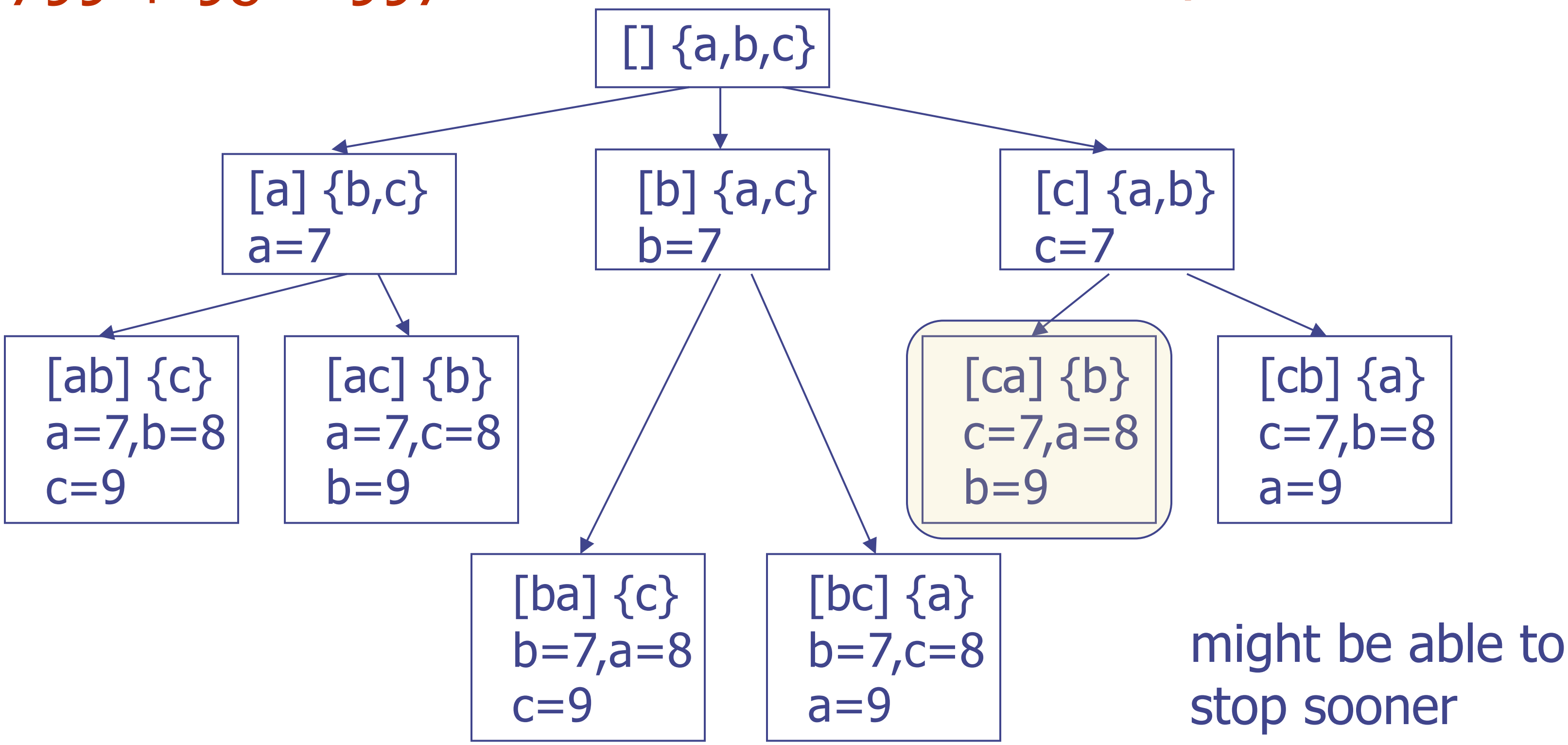
Example

cbb + ba = abc

799 + 98 = 997

a,b,c stand for 7,8,9; not necessarily in that order



Visualizing PuzzleSolve