Ch. 2

Σ = alphabet = finite non-empty set of symbols

Σ^* = set of all strings over Σ

Theorem: Σ^* is countably infinite

Proof sketch: $\Sigma = \{a, b, c\}$

$\Sigma^* = \{\epsilon, a, b, c, aa, ab, ac, ba, bb, bc,$
$ca, cb, cc, aaa, aab, aac, \ldots$

Theorem: $\mathcal{L}$ = set of all languages over Σ is uncountab

Proof: Diagonalization argument

Regular languages

Def: It is defined as follows:

1) $\{a\}$ is regular for each $a \in \Sigma$
2) $\{\epsilon\}$ is regular
3) $\{\}$ is regular
4) If L_1 and L_2 are regular then
   $L_1 L_2$ is regular ($L_2 L_1$ is also reguk
   $L_1 \cup L_2$ is regular
   $L_1{}^*$ is regular ($L_2{}^*$ is regular)
5) Nothing else is regular

Reg expressions over Σ

ϕ is a r.e

ϵ " " "

a " " " for each $a \in \Sigma$

$r+s$, rs, r^*, (r) are r.e.

language r

def

8/29/2024

w^+
L^+ concatenate 1 or more times

$$L^+ = L \cdot L^* = L \cdot (\{\epsilon\} \cup L \cup L^2 \cup L^3 \cdots\cdots)$$

recall $L^0 = \{\epsilon\}$ $= L\{\epsilon\} \cup \underline{L \cdot L} \cup \underline{L \cdot L^2} \cdots$

Regular Languages over Σ

(1) $\{\}$ is regular
(2) $\{\epsilon\}$ is regular
(3) $\{a\}$ is regular, for each $a \in \Sigma$

(4) if L_1 and L_2 are regular languages then so are
$L_1 \cdot L_2$, $L_1 \cup L_2$, L_1^*

(5) Nothing else is regular

Regular Expression over Σ

(compact notation to <u>denote</u> reg. lang.)
(finite representations infinite languages)

(1) ϕ is a reg expr
(2) ϵ is a reg expr
(3) a is a reg expr, for each $a \in \Sigma$
(4) if r and s are reg expr then so are
rs, $r+s$, r^*, (r)
(5) Nothing else is a reg. expr.

def the language denoted by reg expr r, $L(r)$, is defined as follows:

$$L(\phi) = \{\ \}$$
$$L(\epsilon) = \{\epsilon\}$$
$$L(a) = \{a\}, \text{ for each } a \in \Sigma$$
$$L(rs) = L(r) \cdot L(s)$$
$$L(r+s) = L(r) \cup L(s)$$
$$L(r^*) = L(r)^*$$

Examples

$\Sigma = \{a, b\}$

$L_1 = \{ abaab, aaaaaa, a \}$

Why?

$L_1 = \{a\}\cdot\{b\}\cdot\{a\}\cdot\{a\}\cdot\{b\} \cup$

$\{a\}\cdot\{a\}\cdot\{a\}\cdot\{a\}\cdot\{a\}\cdot\{a\} \cup$

$\{a\}$

or $L_1 = L(abaab + aaaaaa + a)$

$L_2 = \{ w \mid w \in \Sigma^* \text{ and } w \text{ ends with 'a'} \}$

$L_2 = L((a+b)^* a)$ $\qquad L((a+b)^*) = \Sigma^*$

ends with 2 a's $\quad (a+b)^* aa$

begins with 2 a's $\quad aa(a+b)^*$

contains 'aab' as substring

$(a+b)^* aab (a+b)^*$

$L_q = \{ w \mid w \in \{a,b\}^* \text{ and } w \text{ has even \# } a\text{'s} \}$

Claim:

$L_q = L\left(b^*(b^*ab^*ab^*)^*\right)$

Proof:

to prove: $L_q \subseteq L(b^*(b^*ab^*ab^*)^*)$

let $w \in L_q$

$w \in L(b^*(b^*ab^*ab^*)^*)$

QED

to prove: $L(b^*(b^*ab^*ab^*)^*) \subseteq L_q$

let $w \in L(b^*(b^*ab^*ab^*)^*)$

⋮

w has even # a's

$w \in L_q$

QED

QED

odd a's

b^*ab^* $\qquad$ $b^*(b^*ab^*ab^*)^*$

$\Sigma = \{a, b, c\}$

ends with 'a' or ends with 'b'

$$(a+b+c)^* a + (a+b+c)^* b$$

even a's ~~and~~ or even b's

$$b^*(b^* a b^* a b^*)^* + a^*(a^* b a^* b a^*)^*$$

even a's and even b's

$$\cancel{aa+bb}\ (ab+ba)^* (aa + bb)^* (ab+ba)^*$$??

$\Sigma = \{a, b, c\}$

does not contain 'ac' as a substring

$$(b+c)^* a (a+b) (b+c)^* + (b+c)^* a$$

??

aa bbbb aa aa aa bb

abba

L_0 = alternates 1's and 0's
$(01)^* + (10)^* + 0(10)^* + 1(01)^*$

1) L_1 = set of words over $\{a,b,c\}$ ~~with substring 'ac'~~ ~~which~~ which start and end with 'a' 01/08/97

2) $\mathcal{L}_2$ = Set of words over $\{a,b,c\}$ with no substring 'ac'

3) Set of words over $\{a,b,c\}$ with atleast one 'ac' as a substring

4) Set of words with even # a's = {bbb, baabbbaba, ...}

$$(b^* a b^* a b^*)^* b^*$$

5) Set of words with odd # a's

$$b^* a b^* (b^* a b^* a b^*)^*$$

6) words which begin with 'a' or end with 'b'

$$a(a \cup b)^* \cup (a \cup b)^* b$$

7) $\mathcal{L}_2$ = words with even # of a's and even # of b's

$$\mathcal{L}_1 = \mathcal{L}[\, aa \cup bb \cup (ab \cup ba)(aa \cup bb)^*(ab \cup ba)]^*$$

<u>Th. 2.2.2</u>

1) $r \cup s = s \cup r$
2) $r \cup \phi = r = \phi \cup r$

11) $r(sr)^* = (rs)^* r$
12) $(r^* s)^* = \varepsilon \cup (r \cup s)^* s$

1. Σ^* is countably infinite

2. $\mathcal{L}$ = Set of all Languages is uncountable.

"diagonalization" proof ; proof by contradictn

let $\mathcal{L}$ be countable

$$
\begin{array}{c|ccccc}
 & s_1 & s_2 & s_3 & \cdots & \\
\hline
L_1 & 1 & 0 & 0 & & \\
L_2 & 0 & 1 & & & \\
L_3 & & & & & \\
\vdots & & & & &
\end{array}
$$

$s_i \in \Sigma^*$

Place 1 if $s_i \in L$

0 otherwise

3. X = set of all real numbers between 0 and 1 + including is uncountable

Let X be countable.

$$
\begin{array}{ll}
r_1 & 0.d_{11}\,d_{12}\,d_{13}\,\cdots\cdots \\
r_2 & 0.d_{21}\,d_{22}\,d_{23}\,\cdots\cdots \\
r_3 & 0.d_{31}\,d_{32}\,d_{33}\,\cdots \\
\vdots & \vdots
\end{array}
$$

Construct real number

$s = 0.d_1 d_2 d_3 \ldots$

where $d_1 \neq d_{11}$

$d_2 \neq d_{22}$

$\vdots$

Is $s = r_i$?