Introduction to Lambda Calculus

York University CSE 3401

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Overview

- Functions
- λ-calculus : λ-notation for functions
- Free and bound variables
- α- equivalence and β-reduction
- Connection to LISP

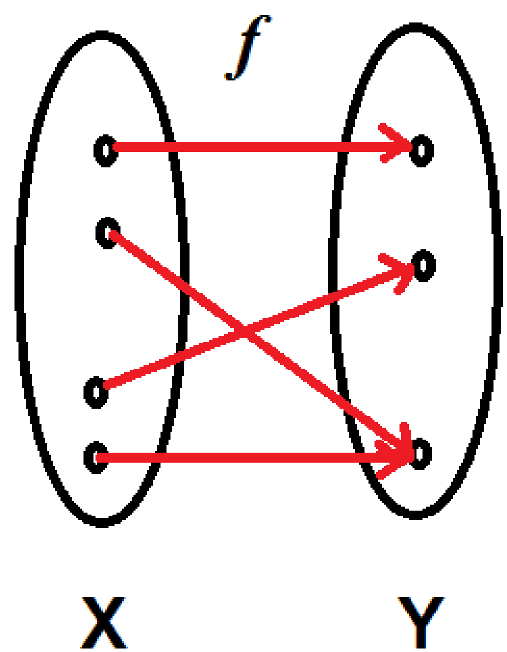
[ref.: Chap. 1 & 2 of Selinger's lecture notes on Lambda Calculus: http://www.mathstat.dal.ca/~selinger/papers/lambdanotes.pdf]

[also Wikipedia on Lambda calculus]

[I am using George Tourlakis' notations for renaming and substitution]

Extensional view of Functions

- "Functions as graphs":
  - each function f has a fixed domain X and co-domain Y
  - a function $f : X \to Y$ is a set of pairs $f \subseteq X \times Y$ such that for each $x \in X$, there exists <u>exactly one</u> $y \in Y$ such that $(x , y) \in f$.



- Equality of functions:
  - Two functions are equal if given the same input they yield the same output

$$f, g : X \to Y, \quad f = g \Leftrightarrow \forall x \in X, f(x) = g(x)$$

Intensional view of Functions

- "Functions as rules":
  - Functions defined as rules, e.g. $f(x) = x^2$
  - Not always necessary to specify domain and co-domain
- Equality of functions:
  - Two functions are equal if they are defined by (essentially) the same formula
- Comparing the two views
  - Graph model is more general, does not need a formula
  - Rule model is more interesting for computer scientists (How can it be calculated? What is the time/memory complexity? etc)

3 observations about functions

f(x)=x is the identity function
g(x)=x is also the identity function

➔ Functions do not need to be explicitly named

➔ Can be expressed as $x \mapsto x$

$(x, y) \mapsto x - y$

$(u, v) \mapsto u - v$

they are the same

➔ The specific choice for argument names is irrelevant

$(x, y) \mapsto x - y$

$x \mapsto (y \mapsto x - y)$

➔ Functions can be re-written in a way to accept only one single input (called **currying**)

Lambda Calculus

- These 3 observations are motivations for a new notation for functions: Lambda notation
- λ-calculus: theory of functions as formulas
- Easier manipulation of functions using expressions
- Examples of λ-notation:
  - The identity function *f(x)=x* is denoted as $\lambda x.x$
  - $\lambda x.x$ is the same as $\lambda y.y$ (called α-equivalence)
  - Function *f* defined as $f : x \mapsto x^2$ is written as $\lambda x.x^2$
  - $f(5)$ is $(\lambda x.x^2)(5)$ and evaluates to 25 (called β-reduction)

More examples

- Evaluate

$$\left(\lambda x.\left((\lambda y.x^2 + y^3)(2)\right)\right)(3)$$

$$= \left(\lambda x.x^2 + 2^3\right)(3) = \left(3^2 + 2^3\right) = 17$$

- Evaluate

$$\left(\lambda x.(\lambda y.x^2 + y^3)\right)(2)(3)$$

$$= \left(\lambda y.2^2 + y^3\right)(3) = \left(2^2 + 3^3\right) = 31$$

Higher order functions

- Higher-order functions are functions whose input and/or output are functions
- They can also be expressed in λ-notation
- Example:
  - $f(x)= x^3$ and $g(x)=(f \circ f)(x)= f^{(2)}(x)=f(f(x))=f(x^3)=(x^3)^3=x^9$
  - $f(x)$ is written as $\lambda x.x^3$
  - $g(x)=f(f(x))$ is written as $\lambda x.f(f(x))$
  - The function defined as $f \mapsto f \circ f$ is denoted as $\lambda f.\lambda x.f(f(x))$

Lambda terms

- *λ-term* calculation:
  1. A ***variable*** is a *λ-term* (for example x, y, ...)
  2. If *M* is a *λ-term* and *x* is a variable, then *(λx.M)* is a *λ-term* (called a ***lambda abstraction***)
  3. If M and N are *λ-terms*, then *(MN)* is a *λ-term* (called an ***application***)

  – Note in *λ-notation* we write *(fx)* instead of *f(x)*

Example: Write the steps in *λ-term* calculation of

$$(\lambda x.(\lambda y.(\lambda z.((xz)(yz)))))$$

$$x,\quad y,\quad z,\quad (xz),\quad (yz),\quad ((xz)(yz)),\quad (\lambda z.((xz)(yz))),$$
$$(\lambda y.(\lambda z.((xz)(yz)))),\quad (\lambda x.(\lambda y.(\lambda z.((xz)(yz)))))$$

Conventions

- Conventions for removing parentheses:
  1. Omit outermost parentheses, e.g. *MN* instead of *(MN)*
  2. Applications are left-associative, omit parentheses when not necessary, e.g. *MNP* means *(MN)P*
  3. Body of abstraction extends to right as far as possible, e.g. $\lambda x.MN$ means $\lambda x.(MN)$
  4. Nested abstractions can be contracted, e.g. $\lambda xy.M$ means $\lambda x.\lambda y.M$

Ex: Write the following with as few parentheses as possible:

$$(\lambda x.(\lambda y.(\lambda z.((xz)(yz))))) \qquad \Rightarrow \lambda xyz.xz(yz)$$

Free and bound variables

- In the term $\lambda x.M$
  - λ is said to bind x in M
  - λx is called a binder
  - x is a bound variable
- In the term $\lambda x.xy$
  - x is a bound variable
  - y is a free variable
- In the term $(\lambda x.xy)(\lambda y.yz)$
  - x is a bound variable
  - z is a free variable
  - y has a free and a bound occurrence
  - Set of free variables $FV=\{y,z\}$

Set of free variables

- FV(M): the set of free variables of a term M
  - $FV(x) = \{x\}$,
  - $FV(\lambda x.M) = FV(M) - \{x\}$
  - $FV(MN) = FV(M) \cup FV(N)$,
- Set of free variables in term M defined as

$$\lambda xy.\big(\big(\lambda z.\lambda v.z(zv)\big)(xy)(zu)\big)$$

is :

$$\begin{aligned} FV(M) &= FV\big(\big(\lambda z.\lambda v.z(zv)\big)(xy)(zu)\big) - \{x,y\} \\ &= \big(FV\big(\lambda z.\lambda v.z(zv)\big) \cup FV(xy) \cup FV(zu)\big) - \{x,y\} \\ &= \big(\big(\{z,v\} - \{z,v\}\big) \cup \{x,y\} \cup \{z,u\}\big) - \{x,y\} \\ &= \{z,u\} \end{aligned}$$

α- equivalence

- $\lambda x.x$ is the same as $\lambda y.y$ (both are identity function)
- $\lambda x.x^2$ is the same as $\lambda z.z^2$
- Renaming bound variables does not change the abstraction
- This is called α-equivalence of lambda terms and is denoted as

$$\lambda x.M =_{\alpha} \lambda y.(M\{x \setminus y\})$$

- Where $M\{x\backslash y\}$ denotes <u>renaming</u> every occurrence of x in M to y (assuming y does not already occur in M)
  - Note x is a bound variable in this definition

Substitution

- Substitution is defined for free variables, substituting a variable with a term.
  - $(\lambda x.xy)[y := M] \quad = \quad \lambda x.xM$
  - $(\lambda x.xy)[y := (uv)] \quad = \quad \lambda x.x(uv)$

- Substitution must be defined to avoid capture
  - $(\lambda x.xy)[y := x] \quad \neq \quad \lambda x.xx$
  - $(\lambda x.xy)[y := x] \quad = \quad (\lambda x'.x'y)[y := x] = \lambda x'.x'x$

  - $(\lambda x.yx)[y := (\lambda z.xz)] \neq \quad \lambda x.(\lambda z.xz)x$
  - $(\lambda x.yx)[y := (\lambda z.xz)] = \quad \lambda x'.(\lambda z.xz)x'$

Substitution (cont.)

- Definition:

$x[x := N]$	$\equiv N$	
$y[x := N]$	$\equiv y$	if $x \neq y$
$(MP)[x := N]$	$\equiv (M[x := N])(P[x := N])$	
$(\lambda x.M)[x := N]$	$\equiv \lambda x.M$	
$(\lambda y.M)[x := N]$	$\equiv \lambda y.(M[x := N])$	if $x \neq y$ and $y \notin FV(N)$
$(\lambda y.M)[x := N]$	$\equiv \lambda y'.(M\{y \backslash y'\}[x := N])$	if $x \neq y, y \in FV(N)$, and y' fresh

Capture case!
Bound variable y is **renamed** to y' to avoid capture of free variable y in N

β-reduction

- β-reduction: the process of evaluating a lambda term by giving value to arguments

  For example:
  - $(\lambda x.x^2)(5) \rightarrow_\beta 25$
  - $(\lambda x.y)(z) \rightarrow_\beta y$

- Definition
  - β-redex: A term of the form $(\lambda x.M)N$ (a lamda abstraction applied to another term)
  - It reduces to M[x:=N]
  - The result is called a reduct
  - β-reduction is applied recursively until there is no more redexes left to reduce
  - A lambda term without any β-redexes is said to be in β-normal form

β-reduction – more examples

- $(\lambda x.y)(\lambda z.zz) \rightarrow_\beta y[x:=(\lambda z.zz)] = y$
- $(\lambda x.y)(\lambda w.w) \rightarrow_\beta y[x:=(\lambda z.zz)] = y$
- $(\lambda w.w)(\lambda w.w) \rightarrow_\beta w[w:=(\lambda w.w)] = (\lambda w.w)$

- $(\lambda x.y)((\lambda z.zz)(\lambda w.w))$

  $\rightarrow_\beta (\lambda x.y)\ (zz\ [z:=(\lambda w.w)]\ ) \rightarrow_\beta (\lambda x.y)\ ((\lambda w.w)\ (\lambda w.w)\ )$

  $\rightarrow_\beta (\lambda x.y)\ (\lambda w.w) \rightarrow_\beta (y\ [x:= (\lambda w.w)]\ ) \rightarrow_\beta y$

- Or $(\lambda x.y)((\lambda z.zz)(\lambda w.w))$

  $\rightarrow_\beta y\ [x:= ((\lambda z.zz)(\lambda w.w))\ ] \rightarrow_\beta y$

Why Lambda Calculus?!

Popular question in 1930's:

"What does it mean for a function *f* to be *computable*?"

- Intuitive computability: A pencil-and-paper method to allow a trained person to calculate f(n) for any given n?

1. **Turing**: A function is computable if and only if it can be computed by the Turing machine.
2. **Gödel**: A function is computable if and only if it is general recursive.
3. **Church**: A function is computable if it can be written as a lambda term.

- It has been proven that all three models are equivalent.
- Are they equivalent to 'intuitive computability'? Cannot be answered!

Lambda Calculus as a Programming Language

- Lambda calculus
  - It can be used to encode programs AND data, such as Booleans and natural numbers
  - It is the simplest possible programming language that is Turing complete
  - ‘Pure LISP’ is equivalent to Lambda Calculus
  - ‘LISP’ is Lambda calculus, plus some additional features such as data types, input/output, etc