Propositional Logic

Syntax

Alphabet: consists of the following types of symbols:

- Truth symbols: 0, 1
- Propositional Symbols: P, Q, R, A, B, C, ...
- Propositional Connectives: $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$, $\neg$
- Parenthesis: (and)

Well-Formed Formulas (wff): A well-formed formula over a given alphabet is a sentence created using the following rules:

1. 0 is a wff; 1 is a wff.
2. Any propositional symbol, P, is a wff.
3. If E1 and E2 are wffs then so are:
    a. $(\neg E1)$
    b. $(E1 \wedge E2)$
    c. $(E1 \vee E2)$
    d. $(E1 \rightarrow E2)$
    e. $(E1 \leftrightarrow E2)$
4. Nothing else is a wff.

Some examples of wffs:

1. 0
2. 1
3. P
4. $(P \wedge Q)$
5. $((\neg P) \wedge (\neg Q))$
6. $((P \rightarrow Q) \leftrightarrow (\neg Q))$
7. $((((P \wedge Q) \rightarrow (R \wedge S)) \vee (P \wedge (\neg Q))) \leftrightarrow ((0 \vee P) \vee (1 \rightarrow Q)))$

Some examples of strings that are not wffs:

1. $P(\neg Q)$
2. $P \wedge Q \rightarrow$
3. $\wedge P \vee Q$
4. $(P \bigtriangledown Q)$ & R
5. $P \wedge Q \rightarrow R \wedge S$
   Note: This would be treated as a wff if rules of precedence for operators are considered; The formula would be $((P \wedge Q) \rightarrow (R \wedge S))$

Question: Why is $((P \rightarrow Q) \leftrightarrow (\neg Q))$ a wff?
Answer: The following steps show how to construct the wff using the syntactic definition of wffs:

1. P is a wff, Rule 2 of wff definition
2. Q is a wff, Rule 2 of wff definition
3. $(P \rightarrow Q)$ is a wff, Rule 3d of wff definition
4. $(\neg Q)$ is a wff, Rule 3a of wff definition
5. $((P \rightarrow Q) \leftrightarrow (\neg Q))$ is a wff, Rule 3e of wff definition

Semantics (Meaning)

Interpretation: An interpretation, I, for a wff E is an assignment of truth values (T or F) to each of the propositional symbols in E.

Example: Consider the wff $((P \wedge Q) \rightarrow R)$. There are 8 different interpretations for the wff as ther are 3 propositional symbols or variables and each can be assigned 2 values. These different interpretations are shown below:

	P	**Q**	**R**
I1	T	T	T
I2	T	T	F
I3	T	F	T
I4	T	F	F
I5	F	T	T
I6	F	T	F
I7	F	F	T
I8	F	F	F

In general, if there are n propositional symbols in a wff, the total number of different interpretations will be 2^n.

Meaning (truth value) of wffs: Let E be a wff and I be an interpretation for E. Then, the *truth value of E under I* is evaulated as follows:

1. The wff 0 has the value F; the wff 1 has the value T.
2. The truth value of a propositional symbol, P, is the same as the truth value assigned to P by I.
3. Let E1 and E2 be two wffs. Then,
   a. $(\neg E1)$ has the value T if E1 has the value F;
      $(\neg E1)$ has the value F if E1 has the value T.
   b. $(E1 \wedge E2)$ has the value T if both E1 and E2 have the value T;
      $(E1 \wedge E2)$ has the value F otherwise.

c. (E1 ∨E2) has the value T if E1 has the value T or E2 has the value T; (E1 ∨E2) has the value F otherwise.
d. (E1 → E2) has the value T if E1 has the value F or E2 have the value T; (E1 → E2) has the value F otherwise (i.e. if E1 has the value T and E2 has the value F).
e. (E1 ↔ E2) has the value T if both E1 and E2 have the same truth value; (E1 ↔ E2) has the value F otherwise.

Part 3. of the above definition can be summarized by the following tables:

E1	**(¬ E1)**
T	F
F	T

E1	**E2**	**(E1 ∧ E2)**	**(E1 ∨E2)**	**(E1 → E2)**	**(E1 ↔ E2)**
T	T	T	T	T	T
T	F	F	T	F	F
F	T	F	T	T	F
F	F	F	F	T	T

Question: What is the truth value of the wff ((P ∧ Q) → R) under the interpretation { P ← T, Q ← F, R ← T }.
Answer: The following steps show the calculation of the truth value.

1. P has the truth value T, Rule 2 meaning of wff
2. Q has the truth value F, Rule 2 meaning of wff
3. (P ∧ Q) has the truth value F, Rule 3b meaning of wff
4. R has the truth value T, Rule 2 meaning of wff
5. ((P ∧ Q) → R) has the truth value T, Rule 3d meaning of wff

Question: What is the truth value of the wff ((P → Q) ↔ ((¬ P) ∨Q)) under the interpretation { P ← T, Q ← F }.
Answer: The following steps show the calculation of the truth value.

1. P has the truth value T, Rule 2 meaning of wff
2. Q has the truth value F, Rule 2 meaning of wff
3. (P → Q) has the truth value F, Rule 3d meaning of wff
4. (¬ P) has the truth value F, Rule 3a meaning of wff
5. ((¬ P) ∨Q) has the truth value F, Rule 3c meaning of wff
6. ((P → Q) ↔ ((¬ P) ∨Q)) has the truth value T, Rule 3e meaning of wff

Some Properties of WFFs

- A wff E is *valid* if it has the value T under every interpretation of E. Valid wffs are often called *tautologies* in propositional logic.
- A wff E is *satisfiable* if it has the value T under some (at least one) interpretation of E.
- A wff E is *contradictory* if it has the value F under every interpretation of E.
- A wff E1 *implies* a wff E2 if for any interpretation I for E1 and E2, if E1 has the value T under I then E2 also has the value T under I. (written as E1 $\Rightarrow$ E2)
- Two wffs E1 and E2 are *equivalent* if for any interpretation I for E1 and E2, both E1 and E2 have the same value under I. (written as E1 $\Leftrightarrow$ E2)

Examples

1. $((P \rightarrow Q) \leftrightarrow ((\neg P) \vee Q))$ is valid.
   The wff has 2 propositional symbols; So, there are 4 possible interpretations:
   a. I1 = { P ← T, Q ← T }
      - $(P \rightarrow Q)$ evaluates to T
      - $(\neg P)$ evaluates to F
      - $((\neg P) \vee Q))$ evaluates to T
      - $((P \rightarrow Q) \leftrightarrow ((\neg P) \vee Q))$ evaluates to T
   b. I2 = { P ← T, Q ← F }
      - $(P \rightarrow Q)$ evaluates to F
      - $(\neg P)$ evaluates to F
      - $((\neg P) \vee Q))$ evaluates to F
      - $((P \rightarrow Q) \leftrightarrow ((\neg P) \vee Q))$ evaluates to T
   c. I3 = { P ← F, Q ← T }
      - $(P \rightarrow Q)$ evaluates to T
      - $(\neg P)$ evaluates to T
      - $((\neg P) \vee Q))$ evaluates to T
      - $((P \rightarrow Q) \leftrightarrow ((\neg P) \vee Q))$ evaluates to T
   d. I4 = { P ← F, Q ← F }
      - $(P \rightarrow Q)$ evaluates to T
      - $(\neg P)$ evaluates to T
      - $((\neg P) \vee Q))$ evaluates to T
      - $((P \rightarrow Q) \leftrightarrow ((\neg P) \vee Q))$ evaluates to T

   We see that in each of the 4 interpretations the wff evaluates to T. So, it is valid.

2. $((P \rightarrow Q) \leftrightarrow (P \vee Q))$ is satisfiable.
   Consider the interpretation I = { P ← T, Q ← T }.
   - P has the value T
   - Q has the value T
   - $(P \rightarrow Q)$ has the value T
   - $(P \vee Q)$ has the value T
   - $((P \rightarrow Q) \leftrightarrow (P \vee Q))$ has the value T

Since the wff evaluates to T under I, we say that the wff is satisfiable.

1. $(\neg (P \rightarrow P))$ is contradictory.
   The wff has 1 propositional symbol; So, there are 2 possible interpretations:
   a. $I1 = \{ P \leftarrow T\}$
      - $(P \rightarrow P)$ evaluates to T
      - $(\neg (P \rightarrow P))$ evaluates to F
   b. $I2 = \{ P \leftarrow F\}$
      - $(P \rightarrow P)$ evaluates to T
      - $(\neg (P \rightarrow P))$ evaluates to F

   We see that in each of the 2 interpretations the wff evaluates to F. So, it is contradictory.

1. $(\neg P) \Rightarrow (P \rightarrow Q)$
   The wff has 2 propositional symbols; So, there are 4 possible interpretations:
   a. $I1 = \{ P \leftarrow T, Q \leftarrow T\}$
      - $(\neg P)$ evaluates to F
        So, we do not have to check the value of $(P \rightarrow Q)$
   b. $I2 = \{ P \leftarrow T, Q \leftarrow F\}$
      - $(\neg P)$ evaluates to F
        So, we do not have to check the value of $(P \rightarrow Q)$
   c. $I3 = \{ P \leftarrow F, Q \leftarrow T\}$
      - $(\neg P)$ evaluates to T
      - $(P \rightarrow Q)$ also evaluates to T
   d. $I4 = \{ P \leftarrow F, Q \leftarrow F\}$
      - $(\neg P)$ evaluates to T
      - $(P \rightarrow Q)$ also evaluates to T

   So, we see that in each interpretation where $(\neg P)$ evaluates to T, $(P \rightarrow Q)$ also evaluates to T. So, $(\neg P) \Rightarrow (P \rightarrow Q)$

2. $(P \rightarrow Q) \Leftrightarrow ((\neg Q)\rightarrow (\neg P))$
   The wff has 2 propositional symbols; So, there are 4 possible interpretations:
   a. $I1 = \{ P \leftarrow T, Q \leftarrow T\}$
      - $(P \rightarrow Q)$ evaluates to T
      - $(\neg Q)$ evaluates to F
      - $(\neg P)$ evaluates to F
      - $((\neg Q) \rightarrow (\neg P))$ evaluates to T
   b. $I2 = \{ P \leftarrow T, Q \leftarrow F\}$
      - $(P \rightarrow Q)$ evaluates to F
      - $(\neg Q)$ evaluates to T
      - $(\neg P)$ evaluates to F
      - $((\neg Q) \rightarrow (\neg P))$ evaluates to F
   c. $I3 = \{ P \leftarrow F, Q \leftarrow T\}$
      - $(P \rightarrow Q)$ evaluates to T

- (¬ Q) evaluates to F
- (¬ P) evaluates to T
- ((¬ Q) → (¬ P)) evaluates to T

d. I4 = { P ← F, Q ← F}

- (P → Q) evaluates to T
- (¬ Q) evaluates to T
- (¬ P) evaluates to T
- ((¬ Q) → (¬ P)) evaluates to T

In all 4 interpretations both (P → Q) and ((¬ Q) → (¬ P)) evaluate to the same truth value. Hence, (P → Q) ⇔ ((¬ Q)→ (¬ P))

Informal Terminology: For two English sentences A and B, we say that A iff B, i.e. A if and only if B, to indicate that (a) A is true if B is true and (b) B is true if A is true.

Remarks: Let E1 and E2 be two wffs. Then,

1. E1 is satisfiable **iff** (¬ E1) is not valid.
2. E1 is contradictory **iff** (¬ E1) is valid.
3. E1 ⇒ E2 **iff** (E1 → E2) is valid
4. E1 ⇔ E2 **iff** (E1 ↔ E2) is valid
5. E1 ⇔ E2 **iff** E1 ⇒ E2 and E2 ⇒ E1

These remarks express all properties of wffs in terms of validity of a given wff.

To establish Validity of wffs:

1. Method 1: Truth Table (wff is valid if true under all interpretations)
   Example: Consider ((¬ (A ∨B) ↔ ((¬ A) ∧ (¬ B)))

A	B		((¬	(A	⊦	B)	↔	((¬	A)	∧	(¬	B)))
T	T		F	T	T	T	T	F	T	F	F	T
T	F		F	T	T	F	T	F	T	F	T	F
F	T		F	F	T	T	T	T	F	F	F	T
F	F		T	F	F	F	T	T	F	T	T	F
							All T					

2. The wff is VALID because we observe all Ts in the main connective (↔) column.
3. Method 2: Proof by Contradiction (Assume wff evaluates to F and then show that some propositional symbol gets assigned both T and F).
   Example: Consider wff ((A → B) → ((¬ B) → (¬ A))). Assume the wff evaluates to F in some interpretation. Then the following reasoning steps apply:

- $(A \rightarrow B)$ must evaluate to T
- $((\neg B) \rightarrow (\neg A))$ must evaluate to F.
- $(\neg B)$ must evaluate to T
- $(\neg A)$ must evaluate to F
- B must evaluate to F
- A must evaluate to T
- So, $(A \rightarrow B)$ must evaluate to F. CONTRADICTION.

So, the wff must be valid.

Proofs and Simplification

Well known logical equivalences (E1-E30) and logical implications (I1-I20). We can use these equivalences to simplify wffs. We can use equivalences and implications in proofs.

Simplification

Example 1: Simplify $((P \rightarrow Q) \wedge ((\neg P) \rightarrow Q))$

	Reason
$((P \rightarrow Q) \wedge ((\neg P) \rightarrow Q))$	Given
$\Leftrightarrow ((\neg P) \vee Q) \wedge ((\neg (\neg P)) \vee Q))$	E12, twice
$\Leftrightarrow ((\neg P) \vee Q) \wedge (P \vee Q))$	E11
$\Leftrightarrow (Q \vee (\neg P)) \wedge (Q \vee P))$	E4
$\Leftrightarrow (Q \vee ((\neg P) \wedge P))$	E10
$\Leftrightarrow (Q \vee (P \wedge (\neg P)))$	E4
$\Leftrightarrow (Q \vee 0)$	E22
$\Leftrightarrow Q$	E19

Example 2: Simplify $(P \rightarrow (\neg (P \wedge (\neg Q)))$

	Reason
$(P \rightarrow (\neg (P \wedge (\neg Q)))$	Given
$\Leftrightarrow ((\neg P) \vee (\neg (P \wedge (\neg Q)))$	E12
$\Leftrightarrow (\neg (P \wedge (P \wedge (\neg Q))))$	E8
$\Leftrightarrow (\neg ((P \wedge P) \wedge (\neg Q)))$	E8
$\Leftrightarrow (\neg (P \wedge (\neg Q)))$	E2
$\Leftrightarrow ((\neg P) \vee (\neg (\neg Q)))$	E8
$\Leftrightarrow ((\neg P) \vee Q)$	E11
$\Leftrightarrow (P \rightarrow Q)$	E12

Formal Proofs

- A *theorem* with hypotheses H1, H2, ..., Hn and conclusion C is true if $(H1 \wedge H2 \wedge \ldots \wedge Hn) \Rightarrow C$
- A *formal proof* (*valid argument*) of a theorem consists of a sequence of wffs ending with C, where each wff may be
    1. one of the hypotheses, or
    2. a known tautology, or
    3. derived from wffs earlier in the sequence via the substitution rule, or
    4. inferred from earlier wffs according to certain logical implications or equivalences.

Example:
Consider the following argument: If I study or if I am a genius, then I will pass the course. If I pass the course, then I will be allowed to take the next course. Therefore, if I am not allowed to take the next course, then I am not a genius.
Let
S: I study.
G: I am a genius.
P: I will pass the course.
A: I will be allowed to take the next course.
Then, the theorem to be proved is

H1: $(S \vee G) \rightarrow P$

H2: $P \rightarrow A$

C: $(\neg A) \rightarrow (\neg G)$

Proof:

Step	Reason
1. $(S \vee G) \rightarrow P$	H1
2. $P \rightarrow A$	H2
3. $(S \vee G) \rightarrow A$	1,2; I8
4. $(\neg (S \vdash G)) \vee A$	3; E12 (substitution)
5. $((\neg S) \wedge (\neg G)) \vee A$	4; E7
6. $((\neg S) \vee A) \wedge ((\neg G) \vee A)$	5; E10
7. $(\neg G) \vee A$	6; I4
8. $(G \rightarrow A)$	7; E12
9. $(\neg A) \rightarrow (\neg G)$	8; E14