Predicate Logic

Predicates and Quantifiers

Statements about objects and relationships between objects can be expressed using the notion of a predicate. For example, the statement "John is 50 years old and is 6 feet tall" can be expressed using a predicate with 3 arguments as follows:

person(john,50,6).

As an example of a relationship, the statement "John is the father of Jim" can be expressed using a predicate with 2 arguments as follows:

father(john,jim)

Using the same predicate, other "father" relationships can be expressed. For example:

father(john,tony), father(tony,dan), etc.

Two new symbols (quantifiers) are introduced in predicate logic to express *universal* and *existential* statements: the forall quantifier, $\forall$ and the exists quantifier, $\exists$.

Syntax

Alphabet: consists of the following types of symbols:

- Truth symbols: 0, 1
- Constant symbols: a, b, c, 10, -2.5, ...
- Variable symbols: X, Y, Z, U, V, ...
- Predicate symbols: p, q, r, s, ... (each predicate symbol has an arity = number of arguments); Predicate symbols with arity 0 are equivalent to propositional symbols.
- Connectives: $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$, $\neg$, $\forall$, $\exists$
- Misc symbols: (,), and ,

Terms constant or variable symbols; used to denote objects.

Well formed formulas (wffs) are constructed as follows:

1. 0 is a wff; 1 is a wff.
2. Atomic Formula: p(t1,...,tn) is an atomic formula where p is a predicate symbol of arity n and t1,...,tn are terms.
3. If E1 and E2 are wffs then so are:
    a. ($\neg$ E1)
    b. (E1 $\wedge$ E2)

   c. $(E1 \vee E2)$
   d. $(E1 \rightarrow E2)$
   e. $(E1 \leftrightarrow E2)$
   f. $(\forall X)(E1)$, where X is a variable symbol.
   g. $(\exists X)(E1)$, where X is a variable symbol.
4. Nothing else is a wff.

Note: In the $(\forall X)(E1)$ and $(\exists X)(E1)$, E1 is called the *scope* of the quantifier.

Bound and Free Occurrences of variables: The occurrence of a variable X is said to be *bound* in wff E if it occurs immediately next to a quantifier (such as the occurence of X in $(\forall X)$ or $(\exists X)$) or appears within the scope of a quantifier (such as an occurrence of X in the E part of $(\forall X)(E)$ or $(\exists X)(E)$). If a variable is bound in E, then it is bound to the *innermost* $(\forall X)$ or $(\exists X)$. A variable occurrence that is not bound is called a free occurrence.

Closed wff is one that has no free variables.

Some examples of universally and existentially quantified formulas

- Consider the domain of all animals and let p(X) stand for "X is a parrot" and u(X) stand for "X is ugly". The statement *All parrots are ugly* can be written in predicate logic as follows:

  $(\forall X)(p(X) \rightarrow u(X))$

  and the statement *There is an ugly parrot* can be written in predicate logic as follows:

  $(\exists X)(p(X) \wedge u(X))$

- Consider the domain of people associated with a university and let s(X) stand for "X is a student", m(X) stand for "X likes music", and i(X) stand for "X is intelligent". Then, the statement *All students are intelligent* is expressed as:

  $(\forall X)(s(X) \rightarrow i(X))$

  and the statement *Some intelligent students like music* is expressed as:

  $(\exists X)(s(X) \wedge m(X) \wedge i(X))$

  The statement *Everyone who likes music is not an intelligent student* is expressed as:

  $(\forall X)(m(X) \rightarrow (s(X) \wedge \neg i(X)))$

and the statement *Every student who likes music is not intelligent* is expressed as:

$(\forall X)((m(X) \wedge s(X)) \rightarrow \neg i(X))$

Semantics

Interpretation: An interpretation I for a wff consists of a domain, D, a set of objects, and assigns values to each of a set of constant, variable, and predicate symbols as follows:

- To each constant symbol a, an element $a_I \in D$.
- To each variable symbol X, an element $X_I \in D$.
- To each predicate symbol p or arity n, a mapping from D^n to {T,F}. Here, D^n is the cartesian product of D with itself n times.

Extended Interpretation: Let I be an interpretation over the domain D. For any variable X and $d \in D$, the extended interpretation for X is denoted by $<X \leftarrow d>°I$, is the interpretation in which the variable X is assigned the object d and all other variables/constants/predicate symbols have the same value as in I.

Note: If I is an in interpretation for $(\forall X)$ E or $(\exists X)$ E, then $<X \leftarrow d>°I$ is an interpretation for E.

Semantic Evaluation of wffs under a given interpretation

Let E be a wff and I be an interpretation for E over the domain D. Then, the *value of E under I* is determined by repeatedly applying the following rules:

1. The value of 0 is F.
2. The value of 1 is T.
3. The value of p(t1,...,tn) is $p_I(value_I(t1),...,value_I(tn))$, where $value_I(ti)$ is the domain element assigned to the term ti by I, and P_I is the meaning assigned to p by I.
4. The values of $(\neg E1)$, $(E1 \wedge E2)$, $(E1 \vee E2)$, $(E1 \rightarrow E2)$, and $(E1 \leftrightarrow E2)$ are interpreted in the same way as in propositional logic.
5. The value of $(\forall X)(E)$ under I is T if for every domain element $d \in D$, the value of E is T under $<x\leftarrow d>°I$ and is F if there exists a domain element $d \in D$ such that the value of E is F under $<x\leftarrow d>°I$
6. The value of $(\exists X)(E)$ under I is T if there exists a domain element $d \in D$ such that the value of E is T under $<x\leftarrow d>°I$ and is F if for every domain element $d \in D$, the value of E is F under $<x\leftarrow d>°I$

Example: Consider the wff $(\exists X)(a(X) \wedge (\forall Y)(b(X,Y) \rightarrow c(Y)))$ and the interpretation over the domain of integers that assigns the following meanings to the predicate symbols: a(d) means "$d > 0$", b(d1,d2) means "$d1 > d2$", and c(d) means "$d \leq 0$".

To evaluate the truth value of the wff, since the outermost quantifier is an existential one, we ask the question: Can we assign a value to X which is greater than 0 such that for every Y value which is less than X, Y is less than or equal to 0? The answer is yes! the value which we can assign to X is 1. It so happens that this is the ONLY value we can assign to X to satisfy the property mentioned. Therefore, the wff evaluates to T under the interpretation.

Now consider a different interpretation over the same domain of integers that assigns the following meanings to the predicate symbols: a(d) means "d is an even integer", b(d1,d2) means "d1 < d2", and c(d) means "d is an odd integer". To evaluate the truth value of the wff we ask the question: Can we assign a value to X which is an even integer such that for every Y value which is greater than X, Y is an odd integer? The answer here is no because no matter what value we choose for X, we will find that there are even integers that are greater than the chosen value for X. Therefore, the wff evaluates to F under the interpretation.

Example: Consider the interpretation I over the domain of non-negative integers that assigns the following meaning to the predicate symbols: p(d1,d2) means "d1+d2 = 0".

1. Consider the wff (∃ X)(∃ Y)p(X,Y). To evaluate the truth value of the wff under I, we ask the question: Can we assign values from the domain to X and Y such that the sum of the values is 0. The answer is yes; the values we can assign are 0 to both X and Y. Therefore, the wff evaluates to T under I.
2. Consider the wff (∃ X)(∀ Y)p(X,Y). To evaluate the truth value of the wff under I, we ask the question: Can we assign a value from the domain to X such that for all values of Y, the X+Y is 0. The answer is no; No matter what value is assigned to X, we would notice that for Y=1, the sum of X and Y would never be 0. Therefore, the wff evaluates to F under I.
3. Consider the wff (∀ X)(∃ Y)p(X,Y). To evaluate the truth value of the wff under I, we ask the question: Is it true that for every value assigned to X, there is a value for Y which adds up to 0? The answer is no; For X=1, we cannot find a value for Y which when added to X gives us 0. Therefore, the wff evaluates to F under I.
4. Consider the wff (∀ X)(∀ Y)p(X,Y). To evaluate the truth value of the wff under I, we ask the question: Is it true that for every pair of values assigned to X and Y, X+Y=0? The answer is obviously no; For X=1 and Y=1, the sum is not 0! Therefore, the wff evaluates to F under I.

Predicate Logic

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1 Rules of Inference

1. **Universal Instantiation**(UI)
   $\frac{(\forall x)P(x)}{P(a)}$ for any constant a (assuming there is a unique constant symbol for each domain element).

2. **Universal generalization**(UG)
   $\frac{P(a)}{(\forall x)P(x)}$ for any arbitrary constant a.

3. **Existential Instantiation**(EI)
   $\frac{(\exists x)P(x)}{P(a)}$ for some constant a for which $P(a)$ is true. Nothing is guaranteed about a.
   Example. Domain: Integers. Say, $P(x)$: x is odd and $Q(x)$: x is even.

1.	$(\exists x)P(x) \wedge (\exists y)Q(y)$	
2.	$(\exists x)P(x)$	
3.	$P(a)$	
4.	$(\exists y)Q(y)$	
5.	$Q(a)$	is incorrect
5.	$Q(b)$	OK

4. **Existential Generalization**(EG)
   $\frac{P(a)}{(\exists x)P(x)}$ for some constant a.

2 Proofs of some theorems

1. Every man has two legs.
   John Smith is a man.
   Therefore, John Smith has to legs.
   Let,
   $M(x)$: x is a man.
   $L(x)$: x has two legs.
   js: John Smith.

H1	:	$(\forall x)(M(x) \to L(x))$
H2	:	$M(js)$
C	:	$L(js)$

Proof:

1.	$(\forall x)(M(x) \to L(x))$	HI
2.	$M(js) \to L(js)$	1,UI
3.	$M(js)$	H2
4.	$L(js)$	2,3 MP(I5)

2. Every microcomputer has a serial port.
   Some microcomputers have a parallel port.
   Therefore, some microcomputers have both a serial and a parallel port.
   Let,
   $M(x)$: x is a microcomputer.
   $S(x)$: x has a serial port.
   $P(x)$: x has a parallel port.

H1	:	$(\forall x)(M(x) \to S(x))$
H2	:	$(\exists x)(M(x) \wedge P(x))$
C	:	$(\exists x)(M(x) \wedge S(x) \wedge P(x))$

Proof:

1.	$(\exists x)(M(x) \wedge P(x))$	H2
2.	$M(a) \wedge P(a)$	1,EI(for some a)
3.	$M(a)$	2 I4(Simplification)
4.	$P(a)$	2 I4(Simplification)
5.	$(\forall x)(M(x) \to S(x))$	H1
6.	$M(a) \to S(a)$	5, UI
7.	$S(a)$	3,6 I5
8.	$M(a) \wedge P(a) \wedge S(a)$	3,4,7 Conjunction
9.	$(\exists x)(M(x) \wedge P(x) \wedge S(x))$	8, EG

3. All rock music is loud.
   Some rock music exists.
   Therefore, some loud music exists.
   Let,
   $R(x)$: x is rock music.
   $L(x)$: x is loud.

H1	:	$(\forall x)(R(x) \to L(x))$
H2	:	$(\exists x)R(x)$
C	:	$(\exists x)L(x)$

Proof:

1.	$(\exists x)R(x)$	H2
2.	$R(a)$	1,EI(for some a)
3.	$(\forall x)(R(x) \to L(x))$	H1
4.	$R(a) \to L(a)$	3, UI
5.	$L(a)$	2,4 I5
6.	$(\exists x)L(x)$	5, EG

4. Every member of the board comes from industry or government.

Everyone from government who has a law degree is in favor of the motion.
John is not from industry, but he does have a law degree.
Therefore, if John is a member of the board then he is in favor of the motion.

Let,
$M(x) : x$ is a member of the board.
$I(x) : x$ is from industry.
$G(x) : x$ is from government.
$L(x) : x$ has a law degree.
j : John.

H1	:	$(\forall x)(M(x) \rightarrow (I(x) \vee G(x)))$
H2	:	$(\forall x)((G(x) \wedge L(x)) \rightarrow F(x))$
H3	:	$\neg I(j)$
H4	:	$L(j)$
C	:	$M(j) \rightarrow F(j)$

This can be transformed into:

H1	:	$(\forall x)(M(x) \rightarrow (I(x) \vee G(x)))$
H2	:	$(\forall x)((G(x) \wedge L(x)) \rightarrow F(x))$
H3	:	$\neg I(j)$
H4	:	$L(j)$
H5	:	$M(j)$
C	:	$F(j)$

Proof:

1.	$(\forall x)(M(x) \rightarrow (I(x) \vee G(x)))$	H1
2.	$M(j) \rightarrow (I(j) \vee G(j))$	1,UI
3.	$M(j)$	H5
4.	$I(j) \vee G(j)$	2,3 I5
5.	$(\neg(\neg I(j))) \vee G(j)$	4, DN E11
6.	$(\neg I(j)) \rightarrow G(j)$	5 Implication E12
7.	$\neg I(j)$	H3
8.	$G(j)$	6,7 I5
9.	$L(j)$	H4
10.	$G(j) \wedge L(j)$	8,9 Conj.
11.	$(\forall x)((G(x) \wedge L(x)) \rightarrow F(x))$	H2
12.	$(G(j) \wedge L(j)) \rightarrow F(j)$	11, UI
13.	$F(j)$	10,12 I5

5. C : $(\forall x)P(x) \rightarrow (\forall x)(P(x) \vee Q(x))$

This can be transformed into:

H1	:	$(\forall x)P(x)$
C	:	$(\forall x)(P(x) \vee Q(x))$

Proof:

1.	$(\forall x)P(x)$	H1
2.	$P(a)$	1, UI(arbitrary a)
3.	$P(a) \vee Q(a)$	2, Addition I3
4.	$(\forall x)(P(x) \vee Q(x))$	3, UG (since a was arbitrarily chosen)

6.
$$\frac{}{\text{C} : (\exists x)(A(x) \wedge B(x)) \rightarrow ((\exists x)A(x) \wedge (\exists x)B(x))}$$

This can be transformed into:

$$\frac{\text{H1} : (\exists x)(A(x) \wedge B(x))}{\text{C} : ((\exists x)A(x) \wedge (\exists x)B(x))}$$

Proof:

1.	$(\exists x)(A(x) \wedge B(x))$	H1
2.	$A(c) \wedge B(c)$	1, EI for some c
3.	$A(c)$	2, I4
4.	$B(c)$	2, I4
5.	$(\exists x)A(x)$	3, EG
6.	$(\exists x)B(x)$	4, EG
7.	$(\exists x)A(x) \wedge (\exists x)B(x)$	5,6 Conj.

7.
$$\frac{}{\text{C} : (\exists x)(\forall y)Q(x,y) \rightarrow (\forall y)(\exists x)Q(x,y)}$$

This can be transformed into:

$$\frac{\text{H1} : (\exists x)(\forall y)Q(x,y)}{\text{C} : (\forall y)(\exists x)Q(x,y)}$$

Proof:

1.	$(\exists x)(\forall y)Q(x,y)$	H1
2.	$(\forall y)Q(a,y)$	1, EI for some a
3.	$Q(a,b)$	2, UI for arbitrary b
4.	$(\exists x)Q(x,b)$	3,4 EG
5.	$(\forall y)(\exists x)Q(x,y)$	4, UG because b was chosen arbitrarily

8.
$$\frac{\text{H1} : P(x) \rightarrow (\exists y)Q(x,y)}{\text{C} : (\exists y)(P(x) \rightarrow Q(x,y))}$$

This can be transformed into:

$$\frac{\text{H1} : P(x) \rightarrow (\exists y)Q(x,y)}{\text{C} : P(x) \rightarrow Q(x,a) \text{ for some } a}$$

This can be transformed into:

$$\frac{\begin{array}{l}\text{H1} : P(x) \rightarrow (\exists y)Q(x,y) \\ \text{H2} : P(x)\end{array}}{\text{C} : Q(x,a)}$$

Proof:

1.	$P(x) \to (\exists y)Q(x,y)$	H1
2.	$P(x)$	H2
3.	$(\exists y)Q(x,y)$	1,2 I5
4.	$Q(x,a)$	3, EI for some a

To prove a wff is not valid provide an interpretation in which the wff is false.
Example: $(\exists x)P(x) \wedge (\exists x)Q(x) \to (\exists x)(P(x) \wedge Q(x))$ is not valid.
Proof:

Domain	:	Intergers.
I	:	$P \leftarrow P_I$ where $P_I(d) : d$ is even.
	:	$Q \leftarrow Q_I$ where $Q_I(d) : d$ is odd.

$(\exists x)P(x)$ is true under I because for $x \leftarrow 4$, $P(x)$ is true.
$(\exists x)Q(x)$ is true under I because for $x \leftarrow 5$, $Q(x)$ is true.
Therefore , $(\exists x)P(x) \wedge (\exists x)Q(x)$ is true under I
but $(\exists x)(P(x) \wedge Q(x))$ is false under I because there is no element d of the domain for which $P_I(d) \wedge Q_I(d)$ is true i.e. d is even and d is odd.

Counter examples:

To prove $(\forall x)P(x)$ is false in an interpretation I we find a domain element d for which $P_I(d)$ is false.
d : counter example.
Domain: Integers.

1. $(\forall x)(x$ is negative$)$
   counter example : $x \leftarrow 5$
2. $(\forall x)(x$ is the sum of even integers$)$
   counter example : $x \leftarrow 9$
3. $(\forall x)(x$ is prime $\to x$ is odd$)$
   counter example : $x \leftarrow 2$
4. $(\forall x)(x$ is prime $\to (-1)^x = -1)$
   counter example : $x \leftarrow 2$
5. $(\forall x)(x$ is prime $\to 2^x - 1$ is prime$)$
   counter example : $x \leftarrow 11$

Example
Show that $(\forall x)(P(x) \vee Q(x)) \to (\forall x)P(x) \vee (\forall x)Q(x)$ is not a valid implication.
i.e. show that $(\forall x)(P(x) \vee Q(x)) \to (\forall x)P(x) \vee (\forall x)Q(x)$ is false under some I.
Proof:

$$
\begin{array}{lcl}
D & : & \mathcal{N} = \{0, 1, 2, \ldots\} \\
I & : & P \leftarrow P_I; P_I(d) : d \text{ is even} \\
 & & Q \leftarrow Q_I; Q_I(d) : d \text{ is odd}
\end{array}
$$

$(\forall x)(P(x) \vee Q(x))$ is true under I because for any $d \in D$, $P_I(d) \vee Q_I(d)$ is true. i.e. "d is even" or "d is odd" is true but
$(\forall x)P(x)$ is false under I
counter example : $x \leftarrow 5$
and
$(\forall x)Q(x)$ is false under I
counter example : $x \leftarrow 4$