Find least, minimal, greatest, and maximal elements in a partially ordered set.

Find the equivalence classes associated with an equivalence relation.

MAIN IDEAS

A binary relation on a set S is formally a subset of $S \times S$; the distinctive relationship satisfied by the relation's members often has a verbal description as well.

Operations on binary relations on a set include union, intersection, and complementation.

Binary relations can have properties of reflexivity, symmetry, transitivity, and antisymmetry.

Finite partially ordered sets can be represented graphically.

An equivalence relation on a set defines equivalence classes, which may themselves be treated as entities. An equivalence relation on a set S determines a partition of S, and conversely.

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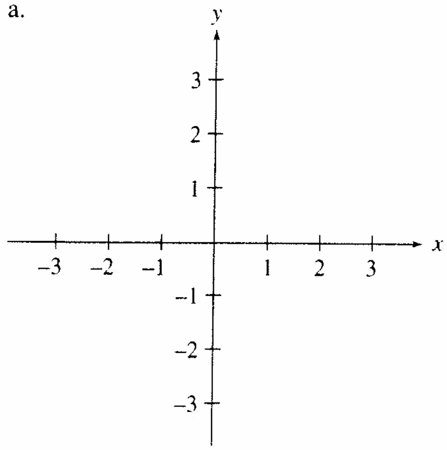
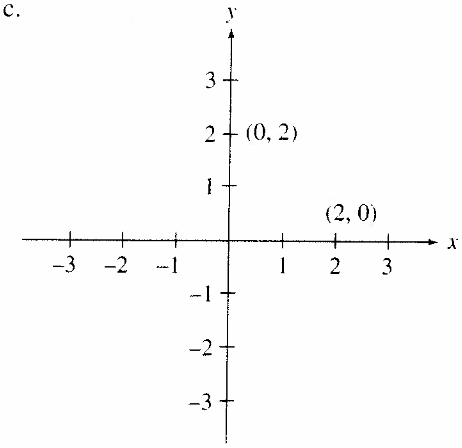
1. For each of the following binary relations ρ on $\mathbb{N}$, decide which of the given ordered pairs belong to ρ.
   a. $x \rho y \leftrightarrow x + y < 7$; (1, 3), (2, 5), (3, 3), (4, 4)
   b. $x \rho y \leftrightarrow x = y + 2$; (0, 2), (4, 2), (6, 3), (5, 3)
   c. $x \rho y \leftrightarrow 2x + 3y = 10$; (5, 0), (2, 2), (3, 1), (1, 3)
   d. $x \rho y \leftrightarrow y$ is a perfect square; (1, 1), (4, 2), (3, 9), (25, 5)
2. For each of the following binary relations ρ on $\mathbb{Z}$, decide which of the given ordered pairs belong to ρ.
   a. $x \rho y \leftrightarrow x|y$; (2, 6), (3, 5), (8, 4), (4, 8)
   b. $x \rho y \leftrightarrow x$ and y are relatively prime; (5, 8), (9, 16), (6, 8), (8, 21)
   c. $x \rho y \leftrightarrow \gcd(x, y) = 7$; (28, 14), (7, 7), (10, 5), (21, 14)
   d. $x \rho y \leftrightarrow x^2 + y^2 = z^2$ for some integer z; (1, 0), (3, 9), (2, 2), (3, 4)
   e. $x \rho y \leftrightarrow x$ is a number from the Fibonacci sequence; (4, 3), (7, 6), (7, 12), (20, 20)
3. Decide which of the given items satisfy the relation.
   a. ρ a binary relation on $\mathbb{Z}$, $x \rho y \leftrightarrow x = -y$; (1, −1), (2, 2), (−3, 3), (−4, −4)
   b. ρ a binary relation on $\mathbb{N}$, $x \rho y \leftrightarrow x$ is prime; (19, 7), (21, 4), (33, 13), (41, 16)
   c. ρ a binary relation on $\mathbb{Q}$, $x \rho y \leftrightarrow x \le 1/y$; (1, 2), (−3, −5), (−4, 1/2), (1/2, 1/3)
   d. ρ a binary relation on $\mathbb{N} \times \mathbb{N}$, $(x, y) \rho (u, v) \leftrightarrow x + u = y + v$; ((1, 2), (3, 2)), ((4, 5), (0, 1))
4. For each of the following binary relations on $\mathbb{R}$, draw a figure to show the region of the plane it describes.
   a. $x \rho y \leftrightarrow y \le 2$
   b. $x \rho y \leftrightarrow x = y - 1$
   c. $x \rho y \leftrightarrow x^2 + y^2 \le 25$
   d. $x \rho y \leftrightarrow x \ge y$

5. For each of the accompanying figures, give the binary relation on $\mathbb{R}$ that describes the shaded area.

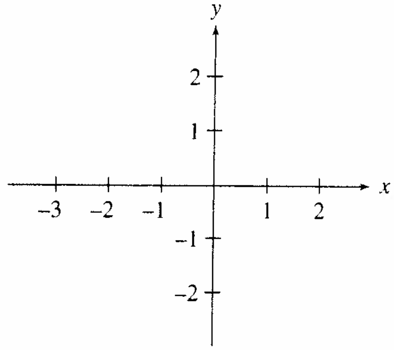
a.



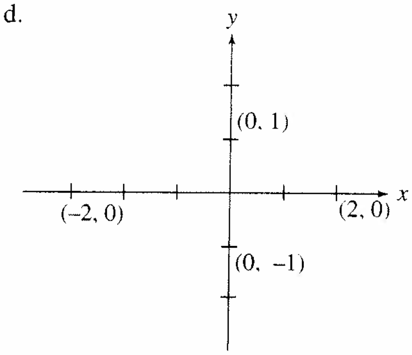
c.



b.



d.



6. Identify each relation on $\mathbb{N}$ as one-to-one, one-to-many, many-to-one, or many-to-many.
   a. $\rho = \{(1, 2), (1, 4), (1, 6), (2, 3), (4, 3)\}$
   b. $\rho = \{(9, 7), (6, 5), (3, 6), (8, 5)\}$
   c. $\rho = \{(12, 5), (8, 4), (6, 3), (7, 12)\}$
   d. $\rho = \{(2, 7), (8, 4), (2, 5), (7, 6), (10, 1)\}$
7. Identify each of the following relations on S as one-to-one, one-to-many, many-to-one, or many-to-many.
   a. $S = \mathbb{N}$
      $x \rho y \leftrightarrow x = y + 1$
   b. $S =$ set of all women in Vicksburg
      $x \rho y \leftrightarrow x$ is the daughter of y
   c. $S = \wp(\{1, 2, 3\})$
      $A \rho B \leftrightarrow |A| = |B|$
   d. $S = \mathbb{R}$
      $x \rho y \leftrightarrow x = 5$
8. Let ρ and σ be binary relations on $\mathbb{N}$ defined by $x \rho y \leftrightarrow$ "x divides y," $x \sigma y \leftrightarrow 5x \le y$. Decide which of the given ordered pairs satisfy the following relations:
   a. $\rho \cup \sigma$; (2, 6), (3, 17), (2, 1), (0, 0)
   b. $\rho \cap \sigma$; (3, 6), (1, 2), (2, 12)
   c. ρ'; (1, 5), (2, 8), (3, 15)
   d. σ'; (1, 1), (2, 10), (4, 8)
9. Let $S = \{1, 2, 3\}$. Test the following binary relations on S for reflexivity, symmetry, antisymmetry, and transitivity.
   a. $\rho = \{(1, 3), (3, 3), (3, 1), (2, 2), (2, 3), (1, 1), (1, 2)\}$
   b. $\rho = \{(1, 1), (3, 3), (2, 2)\}$
   c. $\rho = \{(1, 1), (1, 2), (2, 3), (3, 1), (1, 3)\}$
   d. $\rho = \{(1, 1), (1, 2), (2, 3), (1, 3)\}$

10. Let $S = \{0, 1, 2, 4, 6\}$. Test the following binary relations on S for reflexivity, symmetry, antisymmetry, and transitivity.
   a. $\rho = \{(0, 0), (1, 1), (2, 2), (4, 4), (6, 6), (0, 1), (1, 2), (2, 4), (4, 6)\}$
   b. $\rho = \{(0, 1), (1, 0), (2, 4), (4, 2), (4, 6), (6, 4)\}$
   c. $\rho = \{(0, 1), (1, 2), (0, 2), (2, 0), (2, 1), (1, 0), (0, 0), (1, 1), (2, 2)\}$
   d. $\rho = \{(0, 0), (1, 1), (2, 2), (4, 4), (6, 6), (4, 6), (6, 4)\}$
   e. $\rho = \varnothing$
11. Let S be the set of people in the United States. Test the following binary relations on S for reflexivity, symmetry, antisymmetry, and transitivity.
   a. $x \rho y \leftrightarrow x$ is at least as tall as y.
   b. $x \rho y \leftrightarrow x$ is taller than y.
   c. $x \rho y \leftrightarrow x$ is the same height as y.
   d. $x \rho y \leftrightarrow x$ is a child of y.
   e. $x \rho y \leftrightarrow x$ is the husband of y.
   f. $x \rho y \leftrightarrow x$ is the spouse of y.
   g. $x \rho y \leftrightarrow x$ has the same parents as y.
   h. $x \rho y \leftrightarrow x$ is the brother of y.
12. Test the following binary relations on the given sets S for reflexivity, symmetry, antisymmetry, and transitivity.
   a. $S = \mathbb{Q}$
      $x \rho y \leftrightarrow |x| \le |y|$
   b. $S = \mathbb{Z}$
      $x \rho y \leftrightarrow x - y$ is an integral multiple of 3
   c. $S = \mathbb{N}$
      $x \rho y \leftrightarrow x \cdot y$ is even
   d. $S = \mathbb{N}$
      $x \rho y \leftrightarrow x$ is odd
   e. S = set of all squares in the plane
      $S_1 \rho S_2 \leftrightarrow$ length of side of S_1 = length of side of S_2
   f. S = set of all finite-length strings of characters
      $x \rho y \leftrightarrow$ number of characters in x = number of characters in y
   g. $S = \{0, 1, 2, 3, 4, 5\}$
      $x \rho y \leftrightarrow x + y = 5$
   h. $S = \wp(\{1, 2, 3, 4, 5, 6, 7, 8, 9\})$
      $A \rho B \leftrightarrow |A| = |B|$
   i. $S = \wp(\{1, 2, 3, 4, 5, 6, 7, 8, 9\})$
      $A \rho B \leftrightarrow |A| \ne |B|$
   j. $S = \mathbb{N} \times \mathbb{N}$
      $(x_1, y_1) \rho (x_2, y_2) \leftrightarrow x_1 \le x_2$ and $y_1 \ge y_2$
13. Which of the binary relations of Exercise 12 are equivalence relations? For each equivalence relation, describe the associated equivalence classes.
14. Test the following binary relations on the given sets S for reflexivity, symmetry, antisymmetry, and transitivity.
   a. $S = \mathbb{Z}$
      $x \rho y \leftrightarrow x + y$ is a multiple of 5
   b. $S = \mathbb{Z}$
      $x \rho y \leftrightarrow x < y$
   c. S = set of all finite-length binary strings
      $x \rho y \leftrightarrow x$ is a prefix of y
   d. S = set of all finite-length binary strings
      $x \rho y \leftrightarrow x$ has the same number of 1s as y
   e. $S = \mathbb{Z}$
      $x \rho y \leftrightarrow x = ky$ for some integer k
   f. $S = \mathbb{Z}$
      $x \rho y \leftrightarrow$ there is a prime number p such that $p|x$ and $p|y$
   g. $S = \wp(\{1, 2, 3, 4, 5, 6, 7, 8, 9\})$
      $A \rho B \leftrightarrow A \cap B = \varnothing$

15. For each case, think of a set S and a binary relation ρ on S (different from any in the examples or problems) satisfying the given conditions.
    a. ρ is reflexive and symmetric but not transitive.
    b. ρ is reflexive and transitive but not symmetric.
    c. ρ is not reflexive or symmetric but is transitive.
    d. ρ is reflexive but neither symmetric nor transitive.
16. Let ρ and σ be binary relations on a set S.
    a. If ρ and σ are reflexive, is $\rho \cup \sigma$ reflexive? Is $\rho \cap \sigma$ reflexive?
    b. If ρ and σ are symmetric, is $\rho \cup \sigma$ symmetric? Is $\rho \cap \sigma$ symmetric?
    c. If ρ and σ are antisymmetric, is $\rho \cup \sigma$ antisymmetric? Is $\rho \cap \sigma$ antisymmetric?
    d. If ρ and σ are transitive, is $\rho \cup \sigma$ transitive? Is $\rho \cap \sigma$ transitive?
17. Find the reflexive, symmetric, and transitive closure of each of the relations in Exercise 10.

★ 18. Given the following binary relation

$$S = \text{set of all cities in the country}$$
$$x \,\rho\, y \leftrightarrow \text{Take-Your-Chance Airlines flies directly from } x \text{ to } y$$

describe in words what the transitive closure relation would be.

19. Two additional properties of a binary relation ρ are defined as follows:

ρ is *irreflexive* means $(\forall x)(x \in S \rightarrow (x, x) \notin \rho)$
ρ is *asymmetric* means $(\forall x)(\forall y)(x \in S \wedge y \in S \wedge (x, y) \in \rho \rightarrow (y, x) \notin \rho)$

★ a. Give an example of a binary relation ρ on set $S = \{1, 2, 3\}$ that is neither reflexive nor irreflexive.
b. Give an example of a binary relation ρ on set $S = \{1, 2, 3\}$ that is neither symmetric nor asymmetric.
c. Prove that if ρ is an asymmetric relation on a set S, then ρ is irreflexive.
d. Prove that if ρ is an irreflexive and transitive relation on a set S, then ρ is asymmetric.
e. Prove that if ρ is a nonempty, symmetric, and transitive relation on a set S, then ρ is not irreflexive.

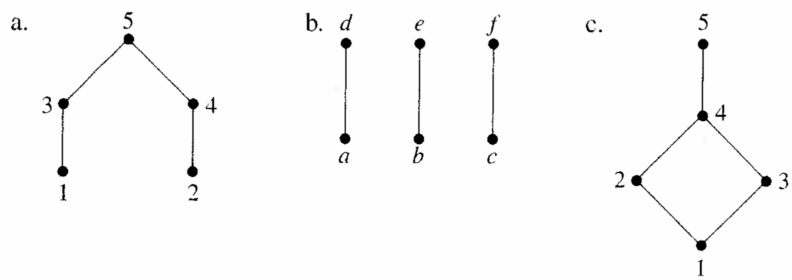
20. Does it make sense to look for the closure of a relation with respect to the following properties? Why or why not?
    a. irreflexive property
    b. asymmetric property
21. Let S be an n-element set. How many different binary relations can be defined on S? (*Hint:* Recall the formal definition of a binary relation.)
22. Let ρ be a binary relation on a set S. For $A \subseteq S$, define

$$\#A = \{x \mid x \in S \wedge (\forall y)(y \in A \rightarrow x \,\rho\, y)\}$$
$$A\# = \{x \mid x \in S \wedge (\forall y)(y \in A \rightarrow y \,\rho\, x)\}$$

a. Prove that if ρ is symmetric, then $\#A = A\#$.
b. Prove that if $A \subseteq B$ then $\#B \subseteq \#A$ and $B\# \subseteq A\#$.
c. Prove that $A \subseteq (\#A)\#$.
d. Prove that $A \subseteq \#(A\#)$.

23. Draw the Hasse diagram for the following partial orderings:
   a. $S = \{a, b, c\}$
      $\rho = \{(a, a), (b, b), (c, c), (a, b), (b, c), (a, c)\}$
   b. $S = \{a, b, c, d\}$
      $\rho = \{(a, a), (b, b), (c, c), (d, d), (a, b), (a, c)\}$
   c. $S = \{\varnothing, \{a\}, \{a, b\}, \{c\}, \{a, c\}, \{b\}\}$
      $A \rho B \leftrightarrow A \subseteq B$
24. For Exercise 23, name any least elements, minimal elements, greatest elements, and maximal elements.
25. Let $(S, \preccurlyeq)$ be a partially ordered set, and let $A \subseteq S$. Prove that the restriction of $\preccurlyeq$ to A is a partial ordering on A.
26. a. Draw the Hasse diagram for the partial ordering "x divides y" on the set $\{2, 3, 5, 7, 21, 42, 105, 210\}$. Name any least elements, minimal elements, greatest elements, and maximal elements. Name a totally ordered subset with four elements.
   b. Draw the Hasse diagram for the partial ordering "x divides y" on the set $\{3, 6, 9, 18, 54, 72, 108, 162\}$. Name any least elements, minimal elements, greatest elements, and maximal elements. Name any unrelated elements.
27. Draw the Hasse diagram for each of the two partially ordered sets.
   a. $S = \{1, 2, 3, 5, 6, 10, 15, 30\}$
      $x \rho y \leftrightarrow x$ divides y
   b. $S = \wp(\{1, 2, 3\})$
      $A \rho B \leftrightarrow A \subseteq B$

   What do you notice about the structure of these two diagrams?
28. For each Hasse diagram of a partial ordering in the accompanying figure, list the ordered pairs that belong to the relation.



29. Let (S, ρ) and (T, σ) be two partially ordered sets. A relation μ on $S \times T$ is defined by $(s_1, t_1)\ \mu\ (s_2, t_2) \leftrightarrow s_1 \rho s_2$ and $t_1 \sigma t_2$. Show that μ is a partial ordering on $S \times T$.
30. Let ρ be a binary relation on a set S. Then a binary relation called the *inverse* of ρ, denoted by ρ^{-1}, is defined by $x \rho^{-1} y \leftrightarrow y \rho x$.
   a. For $\rho = \{(1, 2), (2, 3), (5, 3), (4, 5)\}$ on the set $\mathbb{N}$, what is ρ^{-1}?
   b. Prove that if ρ is a reflexive relation on a set S, then ρ^{-1} is reflexive.
   c. Prove that if ρ is a symmetric relation on a set S, then ρ^{-1} is symmetric.
   d. Prove that if ρ is an antisymmetric relation on a set S, then ρ^{-1} is antisymmetric.
   e. Prove that if ρ is a transitive relation on a set S, then ρ^{-1} is transitive.
   f. Prove that if ρ is an irreflexive relation on a set S (see Exercise 19), then ρ^{-1} is irreflexive.
   g. Prove that if ρ is an asymmetric relation on a set S (see Exercise 19), then ρ^{-1} is asymmetric.

MAIN IDEAS

- PERT charts are diagrams of partially ordered sets representing tasks and prerequisites among tasks.
- A topological sort extends a partial ordering on a finite set to a total ordering.

1. The following tasks are required in order to assemble a bicycle. As the manufacturer, you must write a list of sequential instructions for the buyer to follow. Will the sequential order given below work? Give another sequence that could be used.

Task	Prerequisite Tasks
1. Tightening frame fittings	None
2. Attaching handle bars to frame	1
3. Attaching gear mechanism	1
4. Mounting tire on wheel assembly	None
5. Attaching wheel assembly to frame	1, 4
6. Installing brake mechanism	2, 3, 5
7. Adding pedals	6
8. Attaching seat	1
9. Adjusting seat height	7, 8

2. Construct a PERT chart from the following task table.

Task	Prerequisite Tasks	Time to Perform
A	*E*	3
B	*C*, *D*	5
C	*A*	2
D	*A*	6
E	None	2
F	*A*, *G*	4
G	*E*	4
H	*B*, *F*	1

3. Construct a PERT chart from the following task table.

Task	Prerequisite Tasks	Time to Perform
1	2	4
2	3	2
3	8	5
4	3	2
5	4, 7	2
6	5	1
7	3	3
8	None	5

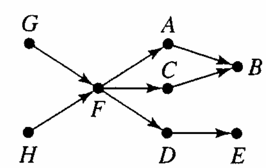
★ 4. Compute the minimum time to completion and the nodes on the critical path for the problem in Exercise 2.

5. Compute the minimum time to completion and the nodes on the critical path for the problem in Exercise 3.

6. For the problem in Exercise 2, great improvements in productivity have knocked down the time to perform task D from 6 units to 1 unit. Recompute the minimum time to completion and the nodes on the critical path.

7. For the problem in Exercise 3, an extra quality-control step has been added to task 4, which now requires 4 units of time to perform. Recompute the minimum time to completion and the nodes on the critical path.

8. Do a topological sort on the partially ordered set shown in the accompanying figure.



★ 9. Find a topological sort for the problem in Exercise 2.

10. Find a topological sort for the problem in Exercise 3.

11. Given the following task chart, find a total ordering in which the tasks can be performed sequentially.

Task	Prerequisite Tasks
1. Chop onions	9
2. Wash lettuce	11
3. Make dressing	11
4. Do stir fry	10
5. Toss salad	2, 3
6. Cut up chicken	None
7. Grate ginger	9
8. Chop bok choy	9
9. Marinate chicken	6
10. Heat wok	1, 7, 8, 11
11. Prepare rice	None

12. A U.S. journalist, on being posted to a bureaucratic foreign country, was faced with the following tasks before she could begin work.

Task	Prerequisite Tasks
1. Obtain a residence permit from the Public Security Bureau	2, 3, 7
2. Obtain a health certificate from the local hospital	None
3. Obtain a journalist work card from the Foreign Ministry	None
4. Obtain a customs certificate from the Customs Office	1, 3, 9
5. Post an announcement in the local newspaper about the presence of her news company in the country	None
6. Obtain a journalist visa from the Public Security Bureau	2, 3, 7
7. Obtain a foreign journalist housing contract from the local housing authority	None
8. Pick up her shipment of belongings from the U.S.	1, 4, 6
9. Obtain a news organization permit from the Foreign Ministry.	5

Find a total ordering in which the tasks can be performed sequentially.