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In Exercises 31–35, $n!$ is the product of the positive integers from 1 to n.

31. Prove that $n! > n^2$ for $n \geq 4$.
32. Prove that $n! > 3^n$ for $n \geq 7$.
33. Prove that $2^n < n!$ for $n \geq 4$.
34. Prove that $2^{n-1} \leq n!$ for $n \geq 1$.
35. Prove that $n! < n^n$ for $n \geq 2$.
36. Prove that $(1 + x)^n > 1 + x^n$ for $n > 1, x > 0$.
37. Prove that $\left(\frac{a}{b}\right)^{n+1} < \left(\frac{a}{b}\right)^n$ for $n \geq 1$ and $0 < a < b$.
38. Prove that $1 + 2 + \cdots + n < n^2$ for $n > 1$.
39. Prove that $1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} < 2 - \frac{1}{n}$ for $n \geq 2$.
40. a. Try to use induction to prove that

$$1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n} < 2 \text{ for } n \geq 1$$

What goes wrong?

b. Prove that

$$1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n} = 2 - \frac{1}{2^n} \text{ for } n \geq 1$$

thus showing that

$$1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n} < 2 \text{ for } n \geq 1$$

41. Prove that $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2^n} \geq 1 + \frac{n}{2}$ for $n \geq 1$. (Note that the denominators increase by 1, not by powers of 2.)

For Exercises 42–53, prove that the statements are true for every positive integer.

42. $2^{3n} - 1$ is divisible by 7.
43. $3^{2n} + 7$ is divisible by 8.
44. $7^n - 2^n$ is divisible by 5.
45. $13^n - 6^n$ is divisible by 7.
46. $2^n + (-1)^{n+1}$ is divisible by 3.
47. $2^{5n+1} + 5^{n+2}$ is divisible by 27.
48. $3^{4n+2} + 5^{2n+1}$ is divisible by 14.
49. $7^{2n} + 16n - 1$ is divisible by 64.
50. $10^n + 3 \cdot 4^{n+2} + 5$ is divisible by 9.
51. $n^3 - n$ is divisible by 3.
52. $n^3 + 2n$ is divisible by 3.
53. $x^n - 1$ is divisible by $x - 1$ for $x \neq 1$.

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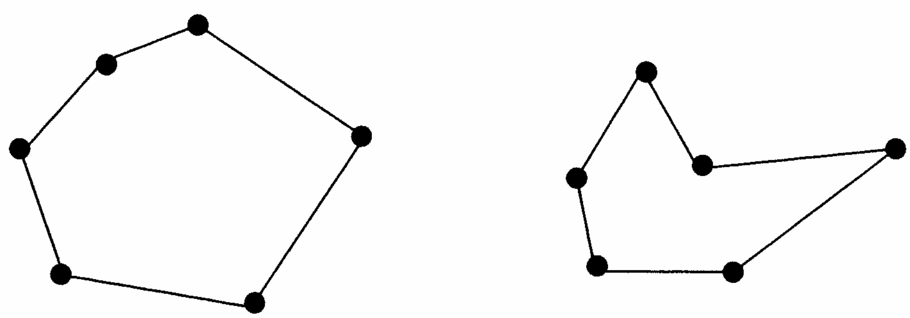
64. What is wrong with the following "proof" by mathematical induction? We will prove that all computers are built by the same manufacturer. In particular, we will prove that in any collection of n computers where n is a positive integer, all of the computers are built by the same manufacturer. We first prove $P(1)$, a trivial process, because in any collection consisting of one computer, there is only one manufacturer. Now we assume $P(k)$; that is, in any collection of k computers, all the computers were built by the same manufacturer. To prove $P(k + 1)$, we consider any collection of $k + 1$ computers. Pull one of these $k + 1$ computers (call it HAL) out of the collection. By our assumption, the remaining k computers all have the same manufacturer. Let HAL change places with one of these k computers. In the new group of k computers, all have the same manufacturer. Thus, HAL's manufacturer is the same one that produced all the other computers, and all $k + 1$ computers have the same manufacturer.
65. An obscure tribe has only three words in its language, *moon*, *noon*, and *soon*. New words are composed by juxtaposing these words in any order, as in *soonnoonmoonnoon*. Any such juxtaposition is a legal word.
    a. Use the first principle of induction (on the number of subwords in the word) to prove that any word in this language has an even number of *o*'s.
    b. Use the second principle of induction (on the number of subwords in the word) to prove that any word in this language has an even number of *o*'s.
66. A *simple closed polygon* consists of n points in the plane joined in pairs by n line segments; each point is the endpoint of exactly two line segments. Following are two examples.



    a. Use the first principle of induction to prove that the sum of the interior angles of an n-sided simple closed polygon is $(n - 2)180°$ for all $n \geq 3$.
    b. Use the second principle of induction to prove that the sum of the interior angles of an n-sided simple closed polygon is $(n - 2)180°$ for all $n \geq 3$.
67. The Computer Science Club is sponsoring a jigsaw puzzle contest. Jigsaw puzzles are assembled by fitting two pieces together to form a small block, adding a single piece to a block to form a bigger block, or fitting two blocks together. Each of these moves is considered a step in the solution. Use the second principle of induction to prove that the number of steps required to assemble an n-piece jigsaw puzzle is $n - 1$.
68. Consider propositional wffs that contain only the connectives $\wedge$, $\vee$, and $\rightarrow$ (no negation) and where wffs must be parenthesized when joined by a logical connective. Count each statement letter, connective, or parenthesis as one symbol. For example, $((A) \wedge (B)) \vee ((C) \wedge (D))$ is such a wff, with 19 symbols. Prove that any such wff has an odd number of symbols.

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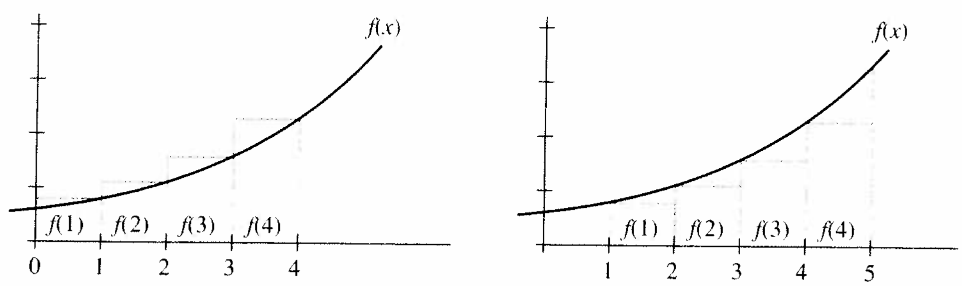
69. Prove that any amount of postage greater than or equal to 2 cents can be built using only 2-cent and 3-cent stamps.
70. Prove that any amount of postage greater than or equal to 12 cents can be built using only 4-cent and 5-cent stamps.
71. Prove that any amount of postage greater than or equal to 14 cents can be built using only 3-cent and 8-cent stamps.
72. Prove that any amount of postage greater than or equal to 64 cents can be built using only 5-cent and 17-cent stamps.
73. Your bank ATM delivers cash using only \$20 and \$50 bills. Prove that you can collect, in addition to \$20, any multiple of \$10 that is \$40 or greater.
74. In any group of k people, $k \geq 1$, each person is to shake hands with every other person. Find a formula for the number of handshakes, and prove the formula using induction.

Exercises 75–76 require familiarity with ideas from calculus. Exercises 1–22 give exact formulas for the sum of terms in a sequence that can be expressed as $\sum_{m=1}^{n} f(m)$. Sometimes it is difficult to find an exact expression for this summation, but if the value of $f(m)$ increases monotonically, integration can be used to find upper and lower bounds on the value of the summation. Specifically,

$$\int_0^n f(x)dx \leq \sum_{m=1}^{n} f(m) \leq \int_1^{n+1} f(x)dx$$

Using the following figure, we can see (on the left) that $\int_0^n f(x)dx$ underestimates the value of the summation while (on the right) $\int_1^{n+1} f(x)dx$ overestimates it.



75. Show that $\int_0^n 2x\, dx \leq \sum_{m=1}^{n} 2m \leq \int_1^{n+1} 2x\, dx$ (see Exercise 2).

76. Show that $\int_0^n x^2\, dx \leq \sum_{m=1}^{n} m^2 \leq \int_1^{n+1} x^2\, dx$ (see Exercise 7).

2.4

Review

TECHNIQUES

Generate values in a sequence defined recursively.

Prove properties of the Fibonacci sequence.

Recognize objects in a recursively defined collection of objects.

Give recursive definitions for particular sets of objects.

Give recursive definitions for certain operations on objects.

Write recursive algorithms to generate sequences defined recursively.

MAIN IDEAS

Recursive definitions can be given for sequences of objects, sets of objects, and operations on objects where basis information is known and new information depends on already known information.

Recursive algorithms provide a natural way to solve certain problems by invoking the same task on a smaller version of the problem.

For Exercises 1–10, write the first five values in the sequence.

1. $S(1) = 10$
   $S(n) = S(n-1) + 10$ for $n \geq 2$
2. $A(1) = 2$
   $A(n) = \dfrac{1}{A(n-1)}$ for $n \geq 2$
3. $B(1) = 1$
   $B(n) = B(n-1) + n^2$ for $n \geq 2$
4. $S(1) = 1$
   $S(n) = S(n-1) + \dfrac{1}{n}$ for $n \geq 2$
5. $T(1) = 1$
   $T(n) = nT(n-1)$ for $n \geq 2$
6. $P(1) = 1$
   $P(n) = n^2P(n-1) + (n-1)$ for $n \geq 2$
7. $M(1) = 2$
   $M(2) = 2$
   $M(n) = 2M(n-1) + M(n-2)$ for $n > 2$
8. $D(1) = 3$
   $D(2) = 5$
   $D(n) = (n-1)D(n-1) + (n-2)D(n-2)$ for $n > 2$
9. $W(1) = 2$
   $W(2) = 3$
   $W(n) = W(n-1)W(n-2)$ for $n > 2$

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10. $T(1) = 1$
$T(2) = 2$
$T(3) = 3$
$T(n) = T(n - 1) + 2T(n - 2) + 3T(n - 3)$ for $n > 3$

In Exercises 11–15, prove the given property of the Fibonacci numbers directly from the definition.

11. $F(n + 1) + F(n - 2) = 2F(n)$ for $n \geq 3$
12. $F(n) = 5F(n - 4) + 3F(n - 5)$ for $n \geq 6$
13. $[F(n + 1)]^2 = [F(n)]^2 + F(n - 1)F(n + 2)$ for $n \geq 2$
14. $F(n + 3) = 2F(n + 1) + F(n)$ for $n \geq 1$
15. $F(n + 6) = 4F(n + 3) + F(n)$ for $n \geq 1$

In Exercises 16–19, prove the given property of the Fibonacci numbers for all $n \geq 1$. (*Hint:* The first principle of induction will work.)

16. $F(1) + F(2) + \cdots + F(n) = F(n + 2) - 1$
17. $F(2) + F(4) + \cdots + F(2n) = F(2n + 1) - 1$
18. $F(1) + F(3) + \cdots + F(2n - 1) = F(2n)$
19. $[F(1)]^2 + [F(2)]^2 + \cdots + [F(n)]^2 = F(n)F(n + 1)$

In Exercises 20–23, prove the given property of the Fibonacci numbers using the second principle of induction.

20. Exercise 14
21. Exercise 15
22. $F(n) < 2^n$ for $n \geq 1$
23. $F(n) > \left(\frac{3}{2}\right)^{n-1}$ for $n \geq 6$
24. The values p and q are defined as follows:

$$p = \frac{1 + \sqrt{5}}{2} \quad \text{and} \quad q = \frac{1 - \sqrt{5}}{2}$$

   a. Prove that $1 + p = p^2$ and $1 + q = q^2$.
   b. Prove that

$$F(n) = \frac{p^n - q^n}{p - q}$$

   c. Use part (b) to prove that

$$F(n) = \frac{\sqrt{5}}{5}\left(\frac{1 + \sqrt{5}}{2}\right)^n - \frac{\sqrt{5}}{5}\left(\frac{1 - \sqrt{5}}{2}\right)^n$$

   is a closed-form solution for the Fibonacci sequence.

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25. A sequence is recursively defined by

$$S(1) = 2$$
$$S(2) = 2$$
$$S(3) = 6$$
$$S(n) = 3S(n-3) \quad \text{for } n \geq 3$$

Prove that $S(n)$ is an even number for $n \geq 1$.

26. A sequence is recursively defined by

$$T(5) = 6$$
$$T(6) = 10$$
$$T(n) = 2T(n-2) + 2 \text{ for } n \geq 7$$

Prove that $T(n) \geq 2n$ for $n \geq 7$.

27. A sequence is recursively defined by

$$S(0) = 1$$
$$S(1) = 1$$
$$S(n) = 2S(n-1) + S(n-2) \text{ for } n \geq 2$$

a. Prove that $S(n)$ is an odd number for $n \geq 0$

b. Prove that $S(n) < 6S(n-2)$ for $n \geq 4$

28. A sequence is recursively defined by

$$T(0) = 1$$
$$T(1) = 2$$
$$T(n) = 2T(n-1) + T(n-2) \text{ for } n \geq 2$$

Prove that $T(n) \leq \left(\frac{5}{2}\right)^n$ for $n \geq 0$.

29. The *Lucas sequence* is defined by

$$L(1) = 1$$
$$L(2) = 3$$
$$L(n) = L(n-1) + L(n-2) \text{ for } n \geq 2$$

a. Write the first five terms of the sequence.

b. Prove that $L(n) = F(n+1) + F(n-1)$ for $n \geq 2$, where F is the Fibonacci sequence.

For Exercises 30–33, decide whether the sequences described are subsequences of the Fibonacci sequence, that is, their members are some or all of the members, in the right order, of the Fibonacci sequence.[2]

30. The sequence $A(n)$, where $A(n) = (n-1)2^{n-2} + 1$, $n \geq 1$. The first four values are 1, 2, 5, 13, which—so far—form a subsequence of the Fibonacci sequence.

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[2]Exercises 30–33 are taken from "Mathematical Recreations" by Ian Stewart, *Scientific American*, May 1995.

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★ 38. In an experiment, a certain colony of bacteria initially has a population of 50,000. A reading is taken every 2 hours, and at the end of every 2-hour interval, there are 3 times as many bacteria as before.

a. Write a recursive definition for $A(n)$, the number of bacteria present at the beginning of the nth time period.

b. At the beginning of which interval are there 1,350,000 bacteria present?

39. An amount of $500 is invested in an account paying 10% interest compounded annually.

a. Write a recursive definition for $P(n)$, the amount in the account at the beginning of the nth year.

b. After how many years will the account balance exceed $700?

40. A collection T of numbers is defined recursively by

1. 2 belongs to T.
2. If X belongs to T, so does $X + 3$ and $2 * X$.

Which of the following belong to T?

a. 6 b. 7 c. 19 d. 12

41. A collection M of numbers is defined recursively by

1. 2 and 3 belong to M.
2. If X and Y belong to M, so does $X * Y$.

Which of the following belong to M?

a. 6 b. 9 c. 16 d. 21 e. 26 f. 54 g. 72 h. 218

★ 42. A collection S of strings of characters is defined recursively by

1. a and b belong to S.
2. If X belongs to S, so does Xb.

Which of the following belong to S?

a. a b. ab c. aba d. $aaab$ e. $bbbbb$

43. A collection W of strings of symbols is defined recursively by

1. a, b, and c belong to W.
2. If X belongs to W, so does $a(X)c$.

Which of the following belong to W?

a. $a(b)c$ b. $a(a(b)c)c$ c. $a(abc)c$ d. $a(a(a(a)c)c)c$ e. $a(aacc)c$

★ 44. Give a recursive definition for the set of all unary predicate wffs in x.

45. Give a recursive definition for the set of all well-formed formulas of integer arithmetic, involving integers together with the arithmetic operations of $+$, $-$, $*$, and $/$.

46. Give a recursive definition for the set of all strings of well-balanced parentheses.

47. Give a recursive definition for the set of all binary strings containing an odd number of 0s.

★ 48. Give a recursive definition for x^R, the reverse of the string x.

49. Give a recursive definition for $|x|$, the length of the string x.

★ 50. Use BNF notation to define the set of positive integers.

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This is a match for Equation (16), where $c = 2$ and $g(n) = 2n$. Therefore $g(2^i) = 2(2^i)$. Substituting into Equation (21)—the solution of Equation (16)—gives the following, where we use the fact that $2^{\log n} = n$.

$$\begin{aligned}
T(n) &= 2^{\log n}\,T(1) + \sum_{i=1}^{\log n} 2^{\log n - i}2(2^i) \\
&= 2^{\log n}(3) + \sum_{i=1}^{\log n} 2^{\log n + 1} \\
&= n(3) + (2^{\log n + 1})\log n \\
&= 3n + (2^{\log n}\cdot 2)\log n \\
&= 3n + 2n\log n
\end{aligned}$$

PRACTICE 25 Show that the solution to the recurrence relation

$$S(1) = 1$$

$$S(n) = 2S\left(\frac{n}{2}\right) + 1 \qquad \text{for } n \geq 2,\ n = 2^m$$

is $2n - 1$. (*Hint:* See Example 15 and note that $2^{\log n} = n$.)

Review

TECHNIQUES

Solve recurrence relations by the expand, guess, and verify technique.

Solve linear, first-order recurrence relations with constant coefficients by using a solution formula.

Solve linear, second-order homogeneous recurrence relations with constant coefficients by using the characteristic equation.

Solve divide-and-conquer recurrence relations by using a solution formula.

MAIN IDEA

Certain recurrence relations have closed-form solutions.

In Exercises 1–9, solve the recurrence relation subject to the basis step.

1. $S(1) = 5$
   $S(n) = S(n - 1) + 5$ for $n \geq 2$
2. $F(1) = 2$
   $F(n) = 2F(n - 1) + 2^n$ for $n \geq 2$

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3. $T(1) = 1$
   $T(n) = 2T(n-1) + 1$ for $n \geq 2$
   (*Hint:* See Example 15.)
4. $A(1) = 1$
   $A(n) = A(n-1) + n$ for $n \geq 2$
   (*Hint:* See Practice 7.)
5. $S(1) = 1$
   $S(n) = S(n-1) + 2n - 1$ for $n \geq 2$
   (*Hint:* See Example 14.)
6. $P(1) = 2$
   $P(n) = 2P(n-1) + n2^n$ for $n \geq 2$
   (*Hint:* See Practice 7.)
7. $F(1) = 1$
   $F(n) = nF(n-1)$ for $n \geq 2$
8. $S(1) = 1$
   $S(n) = nS(n-1) + n!$ for $n \geq 2$
9. $P(1) = 2$
   $P(n) = 3(n+1)P(n-1)$ for $n \geq 2$
10. At the beginning of this chapter the contractor claimed:

    [illegible]

    Write and solve a recurrence relation to check the contractor's claim; note that the end of 20 years is the beginning of the 21st year.
11. A colony of bats is counted every two months. The first four counts are 1200, 1800, 2700, and 4050. If this growth rate continues, what will the 12th count be? (*Hint:* Write and solve a recurrence relation.)
12. Spam e-mail containing a virus is sent to 1000 e-mail addresses. After 1 second, a recipient machine broadcasts 10 new spam e-mails containing the virus, after which the virus disables itself on that machine. How many e-mails are sent at the end of 20 seconds?
13. In an account that pays 8% annually, $1000 is deposited. At the end of each year, an additional $100 is deposited into the account. What is the account worth at the end of 7 years (that is, at the beginning of the 8th year)? (*Hint:* Write and solve a recurrence relation. Also consult Exercise 23 of Section 2.2 for the formula for the sum of a geometric sequence.)
14. A loan of $5000 is charged a 12% annual interest rate. An $80 payment is made each month. How much is left of the loan balance at the end of 18 months (that is, at the beginning of the 19th month)? (*Hint:* Write and solve a recurrence relation. Also consult Exercise 23 of Section 2.2 for the formula for the sum of a geometric sequence).
15. The shellfish population in a bay is estimated to have a count of about 1,000,000. Studies show that pollution reduces this population by about 2% per year, while other hazards are judged to reduce the population by about 10,000 per year. After 9 years, that is, at the beginning of year 10, what is the approximate shellfish population? (*Hint:* Write and solve a recurrence relation. Also consult Exercise 23 of Section 2.2 for the formula for the sum of a geometric sequence.)