15. The sum of an even integer and an odd integer is odd.
16. An odd integer minus an even integer is odd.
17. The product of any two consecutive integers is even.
18. The sum of an integer and its square is even.
★ 19. The square of an even number is divisible by 4.
20. For every integer n, the number

$$3(n^2 + 2n + 3) - 2n^2$$

is a perfect square.
21. If a number x is positive, so is $x + 1$ (do a proof by contraposition).
22. The number n is an odd integer if and only if $3n + 5 = 6k + 8$ for some integer k.
23. The number n is an even integer if and only if $3n + 2 = 6k + 2$ for some integer k.
★ 24. For x and y positive numbers, $x < y$ if and only if $x^2 < y^2$.
25. If $x^2 + 2x - 3 = 0$, then $x \neq 2$.
26. If n is an even prime number, then $n = 2$.
★ 27. If two integers are each divisible by some integer n, then their sum is divisible by n.
28. If the product of two integers is not divisible by an integer n, then neither integer is divisible by n.
29. If n, m, and p are integers and $n|m$ and $m|p$, then $n|p$.
30. If n, m, p, and q are integers and $n|p$ and $m|q$, then $nm|pq$.
31. The sum of three consecutive integers is divisible by 3.
★ 32. The square of an odd integer equals $8k + 1$ for some integer k.
33. The difference of two consecutive cubes is odd.
34. The sum of the squares of two odd integers cannot be a perfect square. (*Hint:* Use Exercise 32.)
★ 35. The product of the squares of two integers is a perfect square.
36. For any two numbers x and y, $|xy| = |x||y|$.
★ 37. For any two numbers x and y, $|x + y| \leq |x| + |y|$.
38. The value A is the average of the n numbers $x_1, x_2, \ldots, x_n$. Prove that at least one of $x_1, x_2, \ldots, x_n$ is greater than or equal to A.
39. Suppose you were to use the steps of Example 11 to attempt to prove that $\sqrt{4}$ is not a rational number. At what point would the proof not be valid?
40. Prove that $\sqrt{3}$ is not a rational number.
41. Prove that $\sqrt{5}$ is not a rational number.
42. Prove that $\sqrt[3]{2}$ is not a rational number.
43. Prove that $\log_2 5$ is not a rational number ($\log_2 5 = x$ means $2^x = 5$).

For Exercises 44–65, prove or disprove the given statement.

44. 91 is a composite number.
★ 45. 297 is a composite number.
46. 83 is a composite number.
47. The difference between two odd integers is odd.
48. The difference between two even integers is even.
★ 49. The product of any three consecutive integers is even.
50. The sum of any three consecutive integers is even.

4. **which**(x: *eat*(x, grass))
5. **which**(x: *eat*(bear, x) **and** *eat*(x, rabbit))
6. **which**(x: *prey*(x) **and** **not**(*eat*(fox, x)))

7. Formulate a Prolog rule that defines "herbivore" to add to the database of Example 39.
8. If the rule of Exercise 7 is included in the database of Example 39, what is the response to the following query?

   which(x: *herbivore*(x))

9. A Prolog database contains the following, where *boss*(x, y) means "x is y's boss" and *supervisor*(x, y) means "x is y's supervisor":

   boss(Mike, Joan)
   boss(Judith, Mike)
   boss(Anita, Judith)
   boss(Judith, Kim)
   boss(Kim, Enrique)
   boss(Anita, Sam)
   boss(Enrique, Jefferson)
   boss(Mike, Hamal)

   supervisor(x, y) if **boss**(x, y)
   supervisor(x, y) if **boss**(x, z) **and** *supervisor*(z, y)

   Find the results of the following queries:

   a. **which**(x: *boss*(x, Sam))
   b. **which**(x: *boss*(Judith, x))
   c. **which**(x: *supervisor*(Anita, x))
10. Declare a Prolog database that gives information about states and capital cities. Some cities are big, others small. Some states are eastern, others are western.
    a. Write a query to find all the small capital cities.
    b. Write a query to find all the states with small capital cities.
    c. Write a query to find all the eastern states with big capital cities.
    d. Formulate a rule to define cosmopolitan cities as big capitals of western states.
    e. Write a query to find all the cosmopolitan cities.
11. Suppose a Prolog database exists that gives information about authors and the books they have written. Books are classified as either fiction, biography, or reference.
    a. Write a query to ask whether Mark Twain wrote *Hound of the Baskervilles*.
    b. Write a query to find all books written by William Faulkner.
    c. Formulate a rule to define nonfiction authors.
    d. Write a query to find all nonfiction authors.
12. Suppose a Prolog database exists that gives information about a family. Predicates of *male*, *female*, and *parent-of* are included.
    a. Formulate a rule to define *father-of*.
    b. Formulate a rule to define *daughter-of*.
    c. Formulate a recursive rule to define *ancestor-of*.

Review

TECHNIQUES

Use the first principle of induction in proofs.

Use the second principle of induction in proofs.

MAIN IDEAS

Mathematical induction is a technique to prove properties of positive integers.

An inductive proof need not begin with 1.

Induction can be used to prove statements about quantities whose values are arbitrary nonnegative integers.

The first and second principles of induction each prove the same conclusion, but one approach may be easier to use than the other in a given situation.

In Exercises 1–22, use mathematical induction to prove that the statements are true for every positive integer n.

1. $2 + 6 + 10 + \cdots + (4n - 2) = 2n^2$
2. $2 + 4 + 6 + \cdots + 2n = n(n + 1)$
3. $1 + 5 + 9 + \cdots + (4n - 3) = n(2n - 1)$
4. $1 + 3 + 6 + \cdots + \dfrac{n(n + 1)}{2} = \dfrac{n(n + 1)(n + 2)}{6}$
5. $4 + 10 + 16 + \cdots + (6n - 2) = n(3n + 1)$
6. $5 + 10 + 15 + \cdots + 5n = \dfrac{5n(n + 1)}{2}$
7. $1^2 + 2^2 + \cdots + n^2 = \dfrac{n(n + 1)(2n + 1)}{6}$
8. $1^3 + 2^3 + \cdots + n^3 = \dfrac{n^2(n + 1)^2}{4}$
9. $1^2 + 3^2 + \cdots + (2n - 1)^2 = \dfrac{n(2n - 1)(2n + 1)}{3}$
10. $1^4 + 2^4 + \cdots + n^4 = \dfrac{n(n + 1)(2n + 1)(3n^2 + 3n - 1)}{30}$
11. $1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + \cdots + n(n + 2) = \dfrac{n(n + 1)(2n + 7)}{6}$
12. $1 + a + a^2 + \cdots + a^{n-1} = \dfrac{a^n - 1}{a - 1}$ for $a \neq 0, a \neq 1$
13. $\dfrac{1}{1 \cdot 2} + \dfrac{1}{2 \cdot 3} + \dfrac{1}{3 \cdot 4} + \cdots + \dfrac{1}{n(n + 1)} = \dfrac{n}{n + 1}$
14. $\dfrac{1}{1 \cdot 3} + \dfrac{1}{3 \cdot 5} + \dfrac{1}{5 \cdot 7} + \cdots + \dfrac{1}{(2n - 1)(2n + 1)} = \dfrac{n}{2n + 1}$