- Use the pigeonhole principle to find the minimum number of elements to guarantee two with a duplicate property.

MAIN IDEA

- The principle of inclusion and exclusion and the pigeonhole principle are additional counting mechanisms for sets.

EXERCISES 3.3

1. In a group of 42 tourists, everyone speaks English or French; there are 35 English speakers and 18 French speakers. How many speak both English and French?

★ 2. All the guests at a dinner party drink coffee or tea; 13 guests drink coffee, 10 drink tea, and 4 drink both coffee and tea. How many people are in this group?

3. After serving 137 customers, a cafeteria notes at the end of the day that 56 orders of green beans were sold, 38 orders of beets were sold, and 17 customers purchased both green beans and beets. How many customers bought neither beans nor beets?

4. Quality control in a factory pulls 40 parts with paint, packaging, or electronics defects from an assembly line. Of these, 28 had a paint defect, 17 had a packaging defect, 13 had an electronics defect, 6 had both paint and packaging defects, 7 had both packaging and electronics defects, and 10 had both paint and electronics defects. Did any part have all three types of defect?

5. In a group of 24 people who like rock, country, and classical music, 14 like rock, 17 like classical, 11 like both rock and country, 9 like rock and classical, 13 like country and classical, and 8 like rock, country, and classical. How many like country?

★ 6. Nineteen different mouthwash products make the following claims: 12 claim to freshen breath, 10 claim to prevent gingivitis, 11 claim to reduce plaque, 6 claim to both freshen breath and reduce plaque, 5 claim to both prevent gingivitis and freshen breath, and 5 claim to both prevent gingivitis and reduce plaque.
   a. How many products make all three claims?
   b. How many products claim to freshen breath but do not claim to reduce plaque?

7. From the 83 students who want to enroll in CS 320, 32 have completed CS 120, 27 have completed CS 180, and 35 have completed CS 215. Of these, 7 have completed both CS 120 and CS 180, 16 have completed CS 180 and CS 215, and 3 have completed CS 120 and CS 215. Two students have completed all three courses. The prerequisite for CS 320 is completion of one of CS 120, CS 180, or CS 215. How many students are not eligible to enroll?

8. Among a bank's 214 customers with checking or savings accounts, 189 have checking accounts, 73 have regular savings accounts, 114 have money market savings accounts, and 69 have both checking and regular savings accounts. No customer is allowed to have both regular savings and money market savings accounts.
   a. How many customers have both checking and money market savings accounts?
   b. How many customers have a checking account but no savings account?

★ 9. A survey of 150 college students reveals that 83 own automobiles, 97 own bikes, 28 own motorcycles, 53 own a car and a bike, 14 own a car and a motorcycle, 7 own a bike and a motorcycle, and 2 own all three.
   a. How many students own a bike and nothing else?
   b. How many students do not own any of the three?

10. At the beginning of this chapter you surveyed the 87 computer users who subscribe to your newsletter in preparation for the release of your new software product.

    The results of your survey reveal that of the 87 subscribers, 68 have a Windows-based system available to them, 34 have a Unix system available, and 30 have access to a Mac. In addition, 19 have access to both Windows and Unix systems, 11 have access to both Unix systems and Macs, and 23 can use both Macs and Windows.

    Use the principle of inclusion and exclusion to determine how many subscribers have access to all three types of systems.
11. You are developing a new bath soap, and you hire a public opinion survey group to do some market research for you. The group claims that in its survey of 450 consumers, the following were named as important factors in purchasing bath soap:

Odor	425
Lathering ease	397
Natural ingredients	340
Odor and lathering ease	284
Odor and natural ingredients	315
Lathering ease and natural ingredients	219
All three factors	147

    Should you have confidence in these results? Why or why not?
12. a. How many integers n, $1 \le n \le 100$, are multiples of either 2 or 5?
    b. How many integers n, $1 \le n \le 100$, are not multiples of either 2 or 5?
13. How many integers n, $1 \le n \le 1000$, are not multiples of either 3 or 7?
14. Write the expression for $|A \cup B \cup C \cup D|$ from equation (4).
★ 15. Write an expression for the number of terms in the expansion of $|A_1 \cup \cdots \cup A_n|$ given by equation (4).
16. Patrons of a local bookstore can sign up for advance notification of new book arrivals in genres of interest. In the first month of this service, 32 sign up for mysteries, 34 for spy novels, 18 for westerns, and 41 for science fiction. Of these, 17 sign for both mysteries and spy novels, 8 for both mysteries and westerns, 19 for mysteries and science fiction, 5 for spy novels and westerns, 20 for spy novels and science fiction, and 12 for westerns and science fiction. In addition, 2 sign up for mysteries, spy novels, and westerns; 11 for mysteries, spy novels, and science fiction; 6 for mysteries, westerns, and science fiction; and 5 for spy novels, westerns, and science fiction. Finally, 2 people sign up for all four categories. How many people signed up for service in the first month?
17. How many cards must be drawn from a standard 52-card deck to guarantee 2 cards of the same suit?
18. How many cards must be drawn from a standard 52-card deck to guarantee a black card?
★ 19. If 12 cards are drawn from a standard deck, must at least 2 of them be of the same denomination?
20. A computerized dating service has a list of 50 men and 50 women. Names are selected at random; how many names must be chosen to guarantee one name of each gender?
21. A computerized housing service has a list of 50 men and 50 women. Names are selected at random; how many names must be chosen to guarantee two names of the same gender?

22. How many people must be in a group in order to guarantee that two people in the group have the same birthday (don't forget leap year)?
23. In a group of 25 people, must there be at least 3 who were born in the same month?
24. Prove that if four numbers are chosen from the set $\{1, 2, 3, 4, 5, 6\}$, at least one pair must add up to 7. (*Hint:* Find all the pairs of numbers from the set that add to 7.)
25. How many numbers must be selected from the set $\{2, 4, 6, 8, 10, 12, 14, 16, 18, 20\}$ to guarantee that at least one pair adds up to 22? (See the hint for Exercise 24).
26. Let n be a positive number. Show that in any set of $n + 1$ numbers, there are at least two with the same remainder when divided by n.

Example 26 in Section 3.2 discussed the problem of counting all possibilities for the last four digits of a telephone number with no repeated digits. In this problem, the number 1259 is not the same as the number 2951 because the order of the four digits is important. An ordered arrangement of objects is called a **permutation**. Each of these numbers is a permutation of 4 distinct objects chosen from a set of 10 distinct objects (the digits). How many such permutations are there? The answer, found by using the multiplication principle, is $10 \cdot 9 \cdot 8 \cdot 7$—there are 10 choices for the first digit, then 9 for the next digit because repetitions are not allowed, 8 for the next digit, and 7 for the fourth digit. The number of permutations of r distinct objects chosen from n distinct objects is denoted by $P(n, r)$. Therefore the solution to the problem of the four-digit number without repeated digits can be expressed as $P(10, 4)$.

A formula for $P(n, r)$ can be written using the factorial function. For a positive integer n, ***n* factorial** is defined as $n(n-1)(n-2)\cdots 1$ and denoted by $n!$; also, $0!$ is defined to have the value 1. From the definition of $n!$, we see that

$$n! = n(n-1)!$$

and that for $r < n$,

$$\begin{aligned}\frac{n!}{(n-r)!} &= \frac{n(n-1)\cdots(n-r+1)(n-r)!}{(n-r)!} \\ &= n(n-1)\cdots(n-r+1)\end{aligned}$$

Using the factorial function,

$$\begin{aligned}P(10, 4) &= 10 \cdot 9 \cdot 8 \cdot 7 \\ &= \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = \frac{10!}{6!} = \frac{10!}{(10-4)!}\end{aligned}$$

In general, $P(n, r)$ is given by the formula

$$P(n, r) = \frac{n!}{(n-r)!} \text{ for } 0 \le r \le n$$

SECTION 3.4 Review

TECHNIQUES

- Find the number of permutations of r distinct objects chosen from n distinct objects.
- Find the number of combinations of r distinct objects chosen from n distinct objects.
- Use permutations and combinations in conjunction with the multiplication principle and the addition principle.
- Find the number of distinct permutations of n objects that are not all distinct.
- Find the number of permutations of r objects out of n distinct objects when objects may be repeated.
- Find the number of combinations of r objects out of n distinct objects when objects may be repeated.
- Generate all permutations of the integers $\{1, \ldots, n\}$ in lexicographical order.
- Generate all combinations of r integers from the set $\{1, \ldots, n\}$.

MAIN IDEAS

- There are formulas for counting various permutations and combinations of objects.
- Care must be taken when analyzing a counting problem to avoid counting the same thing more than once or not counting some things at all.
- Algorithms exist to generate all permutations of n objects and all combinations of r out of n objects.

EXERCISES 3.4

1. Compute the value of the following expressions.
   ★ a. $P(7, 2)$ b. $P(8, 5)$ c. $P(6, 4)$ d. $P(n, n - 1)$
2. How many batting orders are possible for a nine-man baseball team?
3. The 14 teams in the local Little League are listed in the newspaper. How many listings are possible?
4. How many permutations of the characters in COMPUTER are there? How many of these end in a vowel?

★ 5. How many distinct permutations of the characters in ERROR are there? (Remember that the various R's cannot be distinguished from one another.)

6. In how many ways can six people be seated in a circle of six chairs? Only relative positions in the circle can be distinguished.

★ 7. In how many ways can first, second, and third prize in a pie-baking contest be given to 15 contestants?

8. a. Stock designations on an exchange are limited to three letters. How many different designations are there?
   b. How many different designations are there if letters cannot be repeated?
9. In how many different ways can you seat 11 men and 8 women in a row?

★ 10. In how many different ways can you seat 11 men and 8 women in a row if the men all sit together and the women all sit together?

11. In how many different ways can you seat 11 men and 8 women in a row if no 2 women are to sit together?
12. In how many different ways can you seat 11 men and 8 women around a circular table? (Only relative positions in the circle can be distinguished.)
13. In how many different ways can you seat 11 men and 8 women around a circular table if no 2 women are to sit together? (Only relative positions in the circle can be distinguished.)
14. Compute the value of the following expressions.
    ★ a. $C(10, 7)$ b. $C(9, 2)$ c. $C(8, 6)$ d. $C(n, n-1)$
15. Compute $C(n, n-1)$. Explain why $C(n, n-1) = C(n, 1)$.

★ 16. Quality control wants to test 25 microprocessor chips from the 300 manufactured each day. In how many ways can this be done?

17. A soccer team carries 18 players on the roster; 11 players make a team. In how many ways can the team be chosen?

★ 18. In how many ways can a jury of 5 men and 7 women be selected from a panel of 17 men and 23 women?

19. In how many ways can a librarian select 4 novels and 3 plays from a collection of 21 novels and 11 plays?

Exercises 20–23 deal with the following situation: Of a company's personnel, 7 work in design, 14 in manufacturing, 4 in testing, 5 in sales, 2 in accounting, and 3 in marketing. A committee of 6 people is to be formed to meet with upper management.

★ 20. In how many ways can the committee be formed if there is to be one member from each department?

21. In how many ways can the committee be formed if there must be exactly two members from manufacturing?
22. In how many ways can the committee be formed if the accounting department is not to be represented and marketing is to have exactly one representative?

★ 23. In how many ways can the committee be formed if manufacturing is to have at least two representatives?

Exercises 24–28 concern a 5-card hand from a standard 52-card deck. A standard deck has 13 cards from each of 4 suits (clubs, diamonds, hearts, spades). The 13 cards have face value 2 through 10, jack, queen, king, or ace. Each face value is a "kind" of card. The jack, queen, and king are "face cards."

24. How many hands contain four queens?

★ 25. How many hands contain three spades and two hearts?

26. How many hands contain all diamonds?
27. How many hands contain cards from all four suits?
28. How many hands consist of all face cards?

Exercises 29–37 concern 5-card poker hands from a standard 52-card deck.

29. How many hands contain one pair (that is, exactly two cards of the same kind)?

★ 30. How many hands contain two pairs (that is, two pairs of two different kinds plus a fifth card of some third kind)?

31. How many hands contain three of a kind (that is, exactly three cards of the same kind plus two other cards that are not a pair)?

32. How many hands contain a straight (that is, five consecutive cards, for example, ace, 2, 3, 4, 5 or 10, jack, queen, king, ace—aces can be either low or high)?
★ 33. How many hands contain a flush (that is, five cards of the same suit)?
34. How many hands contain a full house (that is, three of a kind plus a pair)?
35. How many hands contain four of a kind?
36. How many hands contain a straight flush (five consecutive cards, for example, ace, 2, 3, 4, 5 of the same suit)?
37. How many hands contain a royal straight flush (that is, the 10, jack, queen, king, ace of one suit)?

For Exercises 38–42, 14 copies of a code module are to be executed in parallel on identical processors organized into two communicating clusters, A and B. Cluster A contains 16 processors and cluster B contains 32 processors.

★ 38. Find the number of ways to choose the processors.
39. Find the number of ways to choose the processors if all modules must execute on cluster B.
40. Find the number of ways to choose the processors if 8 modules are to be processed on cluster A and 6 on cluster B.
41. Find the number of ways to choose the processors if cluster A has 3 failed processors and cluster B has 2 failed processors.
★ 42. Find the number of ways to choose the processors if exactly two modules are to execute on cluster B.

For Exercises 43–46, a set of four coins is selected from a box containing five dimes and seven quarters.

★ 43. Find the number of sets of four coins.
44. Find the number of sets in which two are dimes and two are quarters.
45. Find the number of sets composed of all dimes or all quarters.
46. Find the number of sets with three or more quarters.

Exercises 47–50 concern a computer network with 60 switching nodes.

47. The network is designed to withstand the failure of any two nodes. In how many ways can such a failure occur?
★ 48. In how many ways can one or two nodes fail?
49. If one node has failed, in how many ways can seven nodes be selected without encountering the failed node?
50. If two nodes have failed, in how many ways can seven nodes be selected to include exactly one failed node?

In Exercises 51–54, a congressional committee of three is to be chosen from a set of five Democrats, three Republicans, and four independents.

★ 51. In how many ways can the committee be chosen?
52. In how many ways can the committee be chosen if it must include at least one independent?
★ 53. In how many ways can the committee be chosen if it cannot include both Democrats and Republicans?

54. In how many ways can the committee be chosen if it must have at least one Democrat and at least one Republican?

In Exercises 55–58, a hostess wishes to invite 6 dinner guests from a list of 14 friends.

★ 55. In how many ways can she choose her guests?

56. In how many ways can she choose her guests if six of them are boring and six of them are interesting, and she wants to have at least one of each?

57. In how many ways can she choose her guests if two of her friends dislike each other and neither will come if the other is present?

58. In how many ways can she choose her guests if two of her friends are very fond of each other and one won't come without the other?

★ 59. Twenty-five people, including Simon and Yuan, are candidates to serve on a committee of five. If the committee must include Simon or Yuan, in how many ways can the committee be selected?

60. A student must select 5 classes for the next semester from among 12, but one of the classes must be either American history or English literature. In how many ways can the student choose classes?

61. In a 5-card hand from a standard 52-card deck, how many ways are there to have exactly 4 aces and exactly 1 club?

62. In a 5-card hand from a standard 52-card deck, how many ways are there to have exactly 3 jacks and exactly 2 hearts?

★ 63.
   a. How many distinct permutations are there of the characters in the word HAWAIIAN?
   b. How many of these must begin with H?

64.
   a. How many distinct permutations are there of the characters in the word APALACHICOLA?
   b. How many of these must have both L's together?

65. A bookstore displays a shelf of five, three, and four copies, respectively, of the top three bestsellers. How many distinguishable arrangements of these books are there if books with the same title are not distinguishable?

66. The United Group for Divisive Action uses secret code words that are permutations of five characters. You learn that there are only 10 code words. What can you say about repeated characters in the code words?

★ 67. At a dinner party for five, a tray of five servings of appetizers is prepared. An appetizer could be escargots, egg rolls, or nachos. How many different trays could the kitchen produce?

68. A florist has roses, carnations, lilies, and snapdragons in stock. How many different bouquets of one dozen flowers can be made?

69. A cheese shop carries a large stock of 34 kinds of cheese. By the end of the day, 48 cheese sales have been made, and the items sold must be restocked. How many different restocking orders are possible?

70. One "game package" consists of 12 bingo cards. How many different game packages are there if there are 15 kinds of cards and repetitions are allowed?

★ 71. A hardware shipping order contains 6 items, where each item is either a gallon of paint, a hammer, or a drill.
   a. How many different shipping orders are possible?
   b. How many different shipping orders are possible if no paint is shipped?
   c. How many different shipping orders are possible if each order must contain at least one gallon of paint, one hammer, and one drill?

72. At a birthday party, a mother prepares a plate of cookies for 8 children. There are plenty of chocolate chip, peanut butter, and oatmeal cookies, but each child gets only one cookie.
   a. How many different plates can be prepared?
   b. How many different plates can be prepared if at least one of each kind of cookie is given out?
   c. How many different plates can be prepared if no one likes oatmeal cookies?
   d. How many different plates can be prepared if two children insist on getting peanut butter?
   e. How many different plates can be prepared if the dog got into the kitchen and there are only two chocolate chip cookies left?
73. On Halloween, 10 identical apples are distributed to 7 children.
   a. How many distributions are possible? (*Hint:* One possible distribution is that child 1 gets 3 apples, child 2 gets 0 apples, child 3 gets 2 apples, child 4 gets 0 apples, child 5 and child 6 get 1 apple each, and child 7 gets 3 apples. Although the problem says apples are distributed to children, think of assigning a child's name to each apple; a child's name can go to more than one apple.)
   b. How many distributions are possible if each child is to receive at least one apple?
74. Eight identical antique pie safes are sold at a furniture auction to three bidders.
   a. In how many ways can this be done? (See the hint for Exercise 73.)
   b. In how many ways can this be done if bidder A gets only one pie safe?
75. How many distinct nonnegative integer solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 = 10$$

where the solution

$$x_1 = 3, x_2 = 1, x_3 = 4, x_4 = 2$$

and the solution

$$x_1 = 4, x_2 = 2, x_3 = 3, x_4 = 1$$

are distinct? (*Hint*: Think of this problem as distributing 10 pennies to 4 children; then see the hint in Exercise 73.)
76. How many distinct nonnegative integer solutions are there to the equation

$$x_1 + x_2 + x_3 = 7$$

in which $x_1 \geq 3$? (See the hint for Exercise 75.)
77. Prove that for $n \geq 2$, $P(n, 1) + P(n, 2) = n^2$. (The proof does not require induction, even though it sounds like a very likely candidate for induction.)
78. Prove that for any n and r with $0 \leq r \leq n$, $C(n, r) = C(n, n - r)$. Explain why this is intuitively true.
79. Prove that for any n and r with $0 \leq r \leq n$, $C(n, 2) = C(r, 2) + C(n - r, 2) + r(n - r)$.
80. Prove the identity

$$C(n, r)C(r, k) = C(n, k)C(n - k, r - k) \text{ for } r \leq n \text{ and } k \leq r.$$

Give a combinatorial argument.

SECTION 3.6 Review

TECHNIQUES

- Use the binomial theorem to expand a binomial.
- Use the binomial theorem to find a particular term in the expansion of a binomial.

MAIN IDEAS

- The binomial theorem provides a formula for expanding a binomial without multiplying it out.
- The coefficients of a binomial raised to a nonnegative integer power are combinations of n items as laid out in row n of Pascal's triangle.

EXERCISES 3.6

1. Expand the expression using the binomial theorem.
   - ★ a. $(a + b)^5$
   - b. $(x + y)^6$
   - ★ c. $(a + 2)^5$
   - d. $(a - 4)^4$
   - ★ e. $(2x + 3y)^3$
   - f. $(3x - 1)^5$
   - g. $(2p - 3q)^4$
   - h. $\left(3x + \frac{1}{2}\right)^5$

In Exercises 2–9, find the indicated term in the expansion.

2. The fourth term in $(a + b)^{10}$
3. The seventh term in $(x - y)^{12}$
★ 4. The sixth term in $(2x - 3)^9$
5. The fifth term in $(3a + 2b)^7$
★ 6. The last term in $(x - 3y)^8$
7. The last term in $(ab + 3x)^6$
★ 8. The third term in $(4x - 2y)^5$
9. The fourth term in $\left(3x - \frac{1}{2}\right)^8$
10. Use the binomial theorem (more than once) to expand $(a + b + c)^3$.
11. Expand $(1 + 0.1)^5$ in order to compute $(1.1)^5$.
★ 12. What is the coefficient of x^3y^4 in the expansion of $(2x - y + 5)^8$?
13. What is the coefficient of $x^5y^2z^2$ in the expansion of $(x + y + 2z)^9$?
14. Prove that

$$C(n + 2, r) = C(n, r) + 2C(n, r - 1) + C(n, r - 2) \text{ for } 2 \le r \le n$$

(*Hint:* Use Pascal's formula.)

15. Prove that

$$C(k, k) + C(k + 1, k) + \cdots + C(n, k) = C(n + 1, k + 1) \text{ for } 0 \le k \le n$$

(*Hint:* Use induction on n for a fixed, arbitrary k, as well as Pascal's formula.)

16. Use the binomial theorem to prove that

$$C(n, 0) - C(n, 1) + C(n, 2) - \cdots + (-1)^n C(n, n) = 0$$